# *Integral models in unramified mixed characteristic (0,2) of hermitian orthogonal Shimura varieties of PEL type, Part I*

Adrian Vasiu, U of Arizona, June 26, 2006 (see math.NT/0307205 for earlier versions)



**Abstract.** Let $(G, \mathcal{X})$ be a Shimura variety of PEL type such that $G_{\mathbf{Q}_2}$ is a split $\mathbf{GSO}_{2n}$ group with $n \geq 2$. We prove the existence of the integral canonical models of $\mathrm{Sh}(G, \mathcal{X})/H_2$ in unramified mixed characteristic $(0, 2)$, where $H_2$ is a hyperspecial subgroup of $G(\mathbf{Q}_2)$.

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**Key Words**: Shimura varieties, integral models, abelian schemes, 2-divisible groups, $F$-crystals, reductive and orthogonal group schemes, and involutions.

**Contents**



## 1. Introduction

Let $k$ be an algebraically closed field of characteristic 2. Let $W(k)$ be the ring of Witt vectors with coefficients in $k$. Let $\sigma$ be the Frobenius automorphism of either $k$ or $W(k)$. Let $(M, \Phi)$ be the (contravariant) Dieudonné module of a 2-divisible group $D$ over $k$. Thus $M$ is a free $W(k)$-module of rank equal to the height $h_D$ of $D$ and $\Phi : M \to M$ is a $\sigma$-linear endomorphism such that we have $2M \subseteq \Phi(M)$. Let $D^{\mathrm{t}}$ be the Cartier dual of $D$.

**1.1. Standard deformation spaces.** The simplest *formal deformation spaces* associated to 2-divisible groups over $k$ are the following three:

**(a)** the formal deformation space $\mathfrak{D}$ of $D$ over $\mathrm{Spf}(W(k))$;

**(b)** the formal deformation space $\mathfrak{D}_-$ of $(D, \lambda_D)$ over $\mathrm{Spf}(W(k))$, where $\lambda_D : D \tilde{\to} D^{\mathrm{t}}$ is an (assumed to exist) isomorphism that is a principal quasi-polarization of $D$;

**(c)** the formal deformation space $\mathfrak{D}_+$ of $(D, b_D)$ over $\mathrm{Spf}(W(k))$, where $b_D : D \tilde{\to} D^{\mathrm{t}}$ is an (assumed to exist) isomorphism such that the perfect, bilinear form $b_M : M \otimes_{W(k)} M \to W(k)$ induced naturally by $b_D$ is symmetric.

Both $\mathfrak{D}$ and $\mathfrak{D}_-$ are formally smooth over $\mathrm{Spf}(W(k))$, cf. Grothendieck–Messing deformation theory (see [28, Chs. 4–5], [22, Cor. 4.8], etc.).

In this paragraph we refer to (c). Let $n := \frac{h_D}{2} \in \mathbf{N}$. It is easy to check that if $(D, b_D)$ has a lift to $\mathrm{Spf}(W(k))$ (equivalently to $\mathrm{Spec}(W(k))$, cf. [11, Lemma 2.4.4]), then $b_M$ modulo $2W(k)$ is alternating. If $b_M$ modulo $2W(k)$ is alternating, then the subgroup scheme of $\mathbf{GL}_{M/2M}$ that fixes $b_M$ modulo $2W(k)$ is an $\mathbf{Sp}_{2n}$ group and thus the dimension of the tangent space of $\mathfrak{D}_+$ is $\frac{n(n+1)}{2}$. This dimension is greater than the dimension $\frac{n(n-1)}{2}$ which is predicted by geometric considerations in characteristic 0. Thus $\mathfrak{D}_+$ is not formally smooth over $\mathrm{Spf}(W(k))$ and moreover the Grothendieck–Messing deformation theory does not provide a good understanding of $\mathfrak{D}_+$ (this is specific to characteristic 2!). Thus in the study of good formal subspaces of $\mathfrak{D}_+$, one encounters a major difficulty. This difficulty splits naturally in three problems (parts) that can be described as follows:

**(i)** determine if $b_M$ modulo $2W(k)$ is or is not alternating;

**(ii)** if $b_M$ modulo $2W(k)$ is alternating, then identify a formally closed subscheme $\mathfrak{D}_{++}$ of $\mathfrak{D}_+$ that is formally smooth over $\mathrm{Spf}(W(k))$ of dimension $\frac{n(n-1)}{2}$;

**(iii)** if $\mathfrak{D}_{++}$ exists, show that it has all the expected geometric interpretations.

The formal deformation spaces $\mathfrak{D}$, $\mathfrak{D}_-$, and $\mathfrak{D}_+$ pertain naturally to the study of integral models in mixed characteristic $(0, 2)$ of the simplest cases of unitary, symplectic, and hermitian orthogonal (respectively) Shimura varieties of PEL type. Shimura varieties of PEL type are moduli spaces of polarized abelian schemes endowed with suitable $\mathbf{Z}$-algebras of endomorphisms and with level structures and this explains the *PEL type* terminology (see [40], [26], [25], and [29]).

Shimura varieties of PEL type are the simplest examples of Shimura varieties of abelian type (see [30]). The understanding of the zeta functions of Shimura varieties of abelian type depends on a good theory of their integral models. Such a theory was obtained in [38] for cases of good reduction with respect to primes of characteristic at least 5. But for refined applications to zeta functions one needs also a good theory in mixed characteristic $(0, p)$, with $p \in \{2, 3\}$. As in future work we will present a complete theory of good reductions for $p = 3$, we report here on recent progress for $p = 2$.

**1.2. Previous results.** Let $\mathbf{Z}_{(2)}$ be the localization of $\mathbf{Z}$ with respect to the prime 2. The previous status of the existence of smooth integral models of quotients of Shimura varieties of PEL type in mixed characteristic $(0, 2)$ can be summarized as follows.

**1.2.1.** Mumford proved the existence of the moduli $\mathbf{Z}_{(2)}$-scheme $\mathcal{A}_{n,1,l}$ that parametrizes principally polarized abelian schemes over $\mathbf{Z}_{(2)}$-schemes which are of relative dimension $n$ and which are endowed with a level-$l$ symplectic similitude structure (see [33, Thms. 7.9 and 7.10]); here $n \in \mathbf{N}$ and $l \in 1 + 2\mathbf{N}$. The proof uses geometric invariant theory and standard deformations of abelian varieties. Artin's algebraization method can recover Mumford's result (see [1], [2], and [15, Ch. I, Subsection 4.11]).

**1.2.2.** Drinfeld constructed good moduli spaces of 2-divisible groups over $k$ of dimension 1 (see [13]). See [31] and [19] for applications of them to compact, unitary Shimura varieties related to $\mathbf{SU}(1,n)$ groups over $\mathbf{R}$ (with $n \geq 1$): they provide proper, smooth integrals models of quotients of simple unitary Shimura varieties over localizations of rings of integers of number fields with respect to arbitrary primes of characteristic 2.

**1.2.3.** Using Mumford's result, Serre–Tate deformation theory (see [28], [22], and [23]), and Grothendieck–Messing deformation theory, in [40] and [26] it is proved the existence of good integral models of quotients of unitary and symplectic Shimura varieties of PEL type in unramified mixed characteristic $(0,2)$. These integral models are finite schemes over $\mathcal{A}_{n,1,l}$, are smooth over $\mathbf{Z}_{(2)}$, and are moduli spaces of principally polarized abelian schemes which are of relative dimension $n$ and which are endowed with suitable $\mathbf{Z}$-algebras of endomorphisms and with level-$l$ symplectic similitude structures.

**1.3. Standard PEL situations.** The goal of this paper and of its subsequent Part II is to complete the proof started by Mumford of the existence of good integral models of Shimura varieties of PEL type in unramified mixed characteristic $(0,2)$. We now introduce the *standard PEL situations* used in [40], [26], and [25].

We start with a symplectic space $(W,\psi)$ over $\mathbf{Q}$ and with an injective map

$$f:(G,\mathcal{X})\hookrightarrow(\mathbf{GSp}(W,\psi),\mathcal{S})$$

of Shimura pairs (see [8], [9], [29, Ch. 1], and [38, Subsection 2.4]). Here the pair $(\mathbf{GSp}(W,\psi),\mathcal{S})$ defines a Siegel modular variety (see [29, Example 1.4]). We identify $G$ with a reductive subgroup of $\mathbf{GSp}(W,\psi)$ via $f$. Let $\mathbf{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_{m\mathbf{C}}$ be the unique two dimensional torus over $\mathbf{R}$ with the property that $\mathbf{S}(\mathbf{R})$ is the (multiplicative) group $\mathbf{G}_{m\mathbf{C}}(\mathbf{C})$ of non-zero complex numbers. We recall that $\mathcal{X}$ is a hermitian symmetric domain whose points are a $G(\mathbf{R})$-conjugacy class of monomorphisms $\mathbf{S}\hookrightarrow G_{\mathbf{R}}$ over $\mathbf{R}$ that define Hodge $\mathbf{Q}$–structures on $W$ of type $\{(-1,0),(0,-1)\}$ and that have either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. Let $E(G,\mathcal{X})$ be the number field that is the reflex field of $(G,\mathcal{X})$ (see [9] and [29]). Let $v$ be a prime of $E(G,\mathcal{X})$ of characteristic 2. Let $O_{(v)}$ be the localization of the ring of integers of $E(G,\mathcal{X})$ with respect to $v$ and let $k(v)$ be the residue field of $v$. Let $\mathbf{A}_f := \widehat{\mathbf{Z}}\otimes_{\mathbf{Z}}\mathbf{Q}$ be the ring of finite adèles. Let $\mathbf{A}_f^{(2)}$ be the ring of finite adèles with the 2-component omitted; we have $\mathbf{A}_f=\mathbf{Q}_2\times\mathbf{A}_f^{(2)}$. Let $O(G)$ be the set of compact, open subgroups of $G(\mathbf{A}_f)$ endowed with the inclusion relation. Let $\mathrm{Sh}(G,\mathcal{X})$ be the *canonical model* over $E(G,\mathcal{X})$ of the complex Shimura variety (see [8, Thm. 4.21 and Cor. 5.7]; see [9, Cor. 2.1.11] for the identity)

$$\mathrm{Sh}(G,\mathcal{X})_{\mathbf{C}} := \mathrm{proj.lim.}_{H\in O(G)}G(\mathbf{Q})\backslash\mathcal{X}\times G(\mathbf{A}_f)/H = G(\mathbf{Q})\backslash\mathcal{X}\times G(\mathbf{A}_f). \tag{1}$$

Let $L$ be a $\mathbf{Z}$-lattice of $W$ such that $\psi$ induces a perfect form $\psi: L\otimes_{\mathbf{Z}}L\to\mathbf{Z}$ i.e., the induced monomorphism $L\hookrightarrow L^* := \mathrm{Hom}(L,\mathbf{Z})$ is onto. Let $L_{(2)} := L\otimes_{\mathbf{Z}}\mathbf{Z}_{(2)}$. Let $G_{\mathbf{Z}_{(2)}}$ be the Zariski closure of $G$ in the reductive group scheme $\mathbf{GSp}(L_{(2)},\psi)$. Let

$G_{\mathbf{Z}_2} := G_{\mathbf{Z}_{(2)}} \times_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$, $K_2 := \mathbf{GSp}(L_{(2)}, \psi)(\mathbf{Z}_2)$, and $H_2 := G(\mathbf{Q}_2) \cap K_2 = G_{\mathbf{Z}_{(2)}}(\mathbf{Z}_2)$. Let

$$\mathcal{B} := \{b \in \mathrm{End}(L_{(2)}) | b \text{ fixed by } G_{\mathbf{Z}_{(2)}}\}.$$

Let $G_1$ be the subgroup of $\mathbf{GSp}(W, \psi)$ that fixes all elements of $\mathcal{B}[\frac{1}{2}]$. Let $\mathcal{I}$ be the involution of $\mathrm{End}(L_{(2)})$ defined by the identity $\psi(b(l_1), l_2) = \psi(l_1, \mathcal{I}(b)(l_2))$, where $b \in \mathrm{End}(L_{(2)})$ and $l_1, l_2 \in L_{(2)}$. As $\mathcal{B} = \mathcal{B}[\frac{1}{2}] \cap \mathrm{End}(L_{(2)})$, we have $\mathcal{I}(\mathcal{B}) = \mathcal{B}$. As the elements of $\mathcal{X}$ fix $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} \mathbf{R}$, the involution $\mathcal{I}$ of $\mathcal{B}$ is positive. Let $\mathbf{F}$ be an algebraic closure of $\mathbf{F}_2$.

We will assume that the following four properties (axioms) hold:[1]

**(i)** the $W(\mathbf{F})$-algebra $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F})$ is a product of matrix $W(\mathbf{F})$-algebras;

**(ii)** the $\mathbf{Q}$–algebra $\mathcal{B}[\frac{1}{2}]$ is $\mathbf{Q}$–simple;

**(iii)** the group $G$ is the identity component of $G_1$;

**(iv)** the flat, affine group scheme $G_{\mathbf{Z}_{(2)}}$ over $\mathbf{Z}_{(2)}$ is *reductive* (i.e., it is smooth and its special fibre is connected and has a trivial unipotent radical).

Assumption (iv) implies that $H_2$ is a hyperspecial subgroup of $G(\mathbf{Q}_2) = G_{\mathbf{Q}_2}(\mathbf{Q}_2)$ (cf. [37, Subsubsection 3.8.1]) and that $v$ is unramified over 2 (cf. [30, Cor. 4.7 (a)]). Let $G^{\mathrm{der}}$ be the derived group of $G$. Assumption (ii) is not really required: it is inserted only to ease the presentation. Due to properties (ii) and (iii), one distinguishes the following three possible (and disjoint) cases (see [25, Section 7]):

**(A)** the group $G^{\mathrm{der}}_{\mathbf{C}}$ is a product of $\mathbf{SL}_n$ groups with $n \geq 2$ and, in the case $n = 2$, the center of $G$ has dimension at least 2;

**(C)** the group $G^{\mathrm{der}}_{\mathbf{C}}$ is a product of $\mathbf{Sp}_{2n}$ groups with $n \geq 1$ and, in the case $n = 1$, the center of $G$ has dimension 1;

**(D)** the group $G^{\mathrm{der}}_{\mathbf{C}}$ is a product of $\mathbf{SO}_{2n}$ groups with $n \geq 2$.

We have $G \neq G_1$ if and only if we are in the case (D) i.e., if and only if $G^{\mathrm{der}}$ is not simply connected (cf. [25, Section 7]). In the case (A) (resp. (C) or (D)), one often says that $\mathrm{Sh}(G, \mathcal{X})$ is a *unitary* (resp. a *symplectic* or a *hermitian orthogonal*) *Shimura variety of PEL type* (cf. the description of the intersection group $G_{\mathbf{R}} \cap \mathbf{Sp}(W \otimes_{\mathbf{Q}} \mathbf{R}, \psi)$ in [36, Subsections 2.6 and 2.7]). We are in the case (D) if and only if $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} \mathbf{R}$ is a product of matrix algebras over the quaternion $\mathbf{R}$-algebra $\mathbf{H}$ (see [36, Subsection 2.1, (type III)]).

We refer to the quadruple $(f, L, v, \mathcal{B})$ as a *standard PEL situation in mixed characteristic* $(0, 2)$. Let $\mathcal{M}$ be the $\mathbf{Z}_{(2)}$-scheme which parameterizes isomorphism classes of principally polarized abelian schemes over $\mathbf{Z}_{(2)}$-schemes that are of relative dimension $\frac{\dim_{\mathbf{Q}}(W)}{2}$ and that are equipped with compatible level-$l$ symplectic similitude structures for all numbers $l \in 1 + 2\mathbf{N}$, cf. Subsubsection 1.2.1. We have a natural action of $\mathbf{GSp}(W, \psi)(\mathbf{A}_f^{(2)})$ on $\mathcal{M}$. These symplectic similitude structures and this action are defined naturally via $(L, \psi)$ (see [38, Subsection 4.1]). We identify

1 One can check that the property (iv) implies the property (i).

$\mathrm{Sh}(G,\mathcal{X})_{\mathbf{C}}/H_2 = G_{\mathbf{Z}_{(2)}}(\mathbf{Z}_{(2)})\backslash\mathcal{X} \times G(\mathbf{A}_f^{(2)})$, cf. (1) and [30, Prop. 4.11]. From this identity and the analogous one for $\mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})_{\mathbf{C}}/K_2$, we get that the natural morphism $\mathrm{Sh}(G,\mathcal{X}) \to \mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})_{E(G,\mathcal{X})}$ of $E(G,\mathcal{X})$-schemes (see [8, Cor. 5.4]) induces a closed embedding (see also [38, Rm. 3.2.14])

$$\mathrm{Sh}(G,\mathcal{X})/H_2 \hookrightarrow \mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})_{E(G,\mathcal{X})}/K_2 = \mathcal{M}_{E(G,\mathcal{X})}.$$

Let $\mathcal{N}$ be the Zariski closure of $\mathrm{Sh}(G,\mathcal{X})/H_2$ in $\mathcal{M}_{O_{(v)}}$. Let $\mathcal{N}^{\mathrm{n}}$ be the normalization of $\mathcal{N}$. Let $\mathcal{N}^{\mathrm{s}}$ be the formally smooth locus of $\mathcal{N}^{\mathrm{n}}$ over $O_{(v)}$. Let $(\mathcal{A},\Lambda)$ be the pull back to $\mathcal{N}$ of the universal principally polarized abelian scheme over $\mathcal{M}$.

For a morphism $y : \mathrm{Spec}(k) \to \mathcal{N}_{W(k)}$ of $W(k)$-schemes let

$$(A,\lambda_A) := y^*((\mathcal{A},\Lambda) \times_{\mathcal{N}} \mathcal{N}_{W(k)}).$$

Let $(M_y,\Phi_y,\lambda_{M_y})$ be the principally quasi-polarized $F$-crystal over $k$ of $(A,\lambda_A)$. Thus $(M_y,\Phi_y)$ is a Dieudonné module over $k$ of rank $\dim_{\mathbf{Q}}(W)$ and $\lambda_{M_y}$ is a perfect, alternating form on $M_y$. Let $M_y^* := \mathrm{Hom}(M_y, W(k))$. We denote also by $\lambda_{M_y}$ the perfect, alternating form on $M_y^*$ induced naturally by $\lambda_{M_y}$. We have a natural action of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$ on $M_y^*$ (see beginning of Section 4).

It is well known that in the cases (A) and (C) we have $\mathcal{N} = \mathcal{N}^{\mathrm{s}} = \mathcal{N}^{\mathrm{n}}$, cf. [40] and [26]. Accordingly, in the whole paper we will assume that we are in the case (D). In this paper we will study triples of the form $(M_y,\Phi_y,\lambda_{M_y})$ and the following sequence

$$\mathcal{N}^{\mathrm{s}} \hookrightarrow \mathcal{N}^{\mathrm{n}} \twoheadrightarrow \mathcal{N} \hookrightarrow \mathcal{M}_{O_{(v)}}$$

of morphisms between $O_{(v)}$-schemes. The main goal of this Part I is to prove the following Main Theorem.

**1.4. Main Theorem.** *Suppose that $G_{\mathbf{Z}_2}$ is a split $\mathbf{GSO}_{2n}$ group scheme with $n \geq 2$ (thus we have $k(v) = \mathbf{F}_2$, cf. [30, Prop. 4.6 and Cor. 4.7 (b)]). Then the following three properties hold:*

**(a)** *there exist isomorphisms $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k) \tilde{\to} M_y^*$ of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$-modules that induce symplectic isomorphisms $(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k),\psi) \tilde{\to} (M_y^*,\lambda_{M_y})$;*

**(b)** *the reduced scheme $\mathcal{N}_{k(v)\mathrm{red}}$ of $\mathcal{N}_{k(v)}$ is regular and formally smooth over $k(v)$;*

**(c)** *we have $\mathcal{N}^{\mathrm{s}} = \mathcal{N}^{\mathrm{n}}$ i.e., the $O_{(v)}$-scheme $\mathcal{N}^{\mathrm{n}}$ is regular and formally smooth.*

The locally compact, totally disconnected topological group $G(\mathbf{A}_f^{(2)})$ acts on $\mathcal{N}^{\mathrm{n}}$ continuously in the sense of [9, Subsubsection 2.7.1]. Thus $\mathcal{N}^{\mathrm{n}}$ is an *integral canonical model* of $\mathrm{Sh}(G,\mathcal{X})/H_2$ over $O_{(v)}$ in the sense of [38, Def. 3.2.3 6)], cf. [38, Example 3.2.9 and Cor. 3.4.4]. Due to [39, Thm. 1.3], as in [38, Rms. 3.2.4 and 3.2.7 4')] we get that $\mathcal{N}^{\mathrm{n}}$ is *the unique* integral canonical model of $\mathrm{Sh}(G,\mathcal{X})/H_2$ over $O_{(v)}$ and thus that $\mathcal{N}^{\mathrm{n}}$ is *the final object* of the category of smooth integral models of $\mathrm{Sh}(G,\mathcal{X})/H_2$ over $O_{(v)}$ (here the word smooth is used as in [29, Def. 2.2]).

Part (b) is a tool toward the proof of (c) and toward the future checking that $\mathcal{N}^{\mathrm{s}}$ is $\mathcal{N}$. If $A$ is an ordinary abelian variety, then a result of Noot implies that $\mathcal{N}^{\mathrm{n}}_{W(k)}$ is regular and formally smooth over $W(k)$ at all points through which $y$ factors (see [34, Cor. 3.8]).

We emphasize that no particular case of the Main Theorem was known before. The next Example describes the simplest cases covered by the Main Theorem.

**1.4.1. Example.** We assume that $\mathcal{B}\otimes_{\mathbf{Z}_{(2)}}\mathbf{R}=\mathbf{H}$, that $\mathcal{B}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2=M_2(\mathbf{Z}_2)$, and that we have $\dim_{\mathbf{Q}}(W)=4n$ for some $n\geq 2$. Thus $\mathcal{B}[\frac{1}{2}]$ is a definite quaternion algebra over $\mathbf{Q}$ that splits at 2 and that has a maximal order $\mathcal{B}$ which is a semisimple $\mathbf{Z}_{(2)}$-algebra; moreover we have a $\mathcal{B}[\frac{1}{2}]$-module $W$ of rank $4n$ such that the non-degenerate alternating form $\psi$ on $W$ defines a positive involution of $\mathcal{B}[\frac{1}{2}]$. As $\mathcal{B}\otimes_{\mathbf{Z}_{(2)}}\mathbf{R}=\mathbf{H}$, we are in the case (D) and the group $G^{\mathrm{der}}_{\mathbf{C}}$ is an $\mathbf{SO}_{2n}$ group. It is well known that $G_{\mathbf{Q}_2}$ is a form of the split $\mathbf{GSO}_{2n}$ group over $\mathbf{Q}_2$. Thus if $G_{\mathbf{Z}_2}$ is a reductive group scheme, then it is either a split or a non-split $\mathbf{GSO}_{2n}$ group scheme. The Main Theorem applies only if $G_{\mathbf{Z}_2}$ is a split $\mathbf{GSO}_{2n}$ group scheme.

If $G_{\mathbf{Q}_2}$ is a split $\mathbf{GSO}_{2n}$ group, then it is easy to see that there exists a $\mathbf{Z}_2$-lattice $L_2$ of $W\otimes_{\mathbf{Q}}\mathbf{Q}_2$ such that $\psi$ induces a perfect, alternating form $\psi: L_2\otimes_{\mathbf{Z}_2}L_2\to\mathbf{Z}_2$ and the Zariski closure of $G_{\mathbf{Q}_2}$ in the reductive group scheme $\mathbf{GSp}(L_2,\psi)$ is a split $\mathbf{GSO}_{2n}$ (and therefore also a split reductive) group scheme. We can choose the $\mathbf{Z}$-lattice $L$ of $W$ such that we have $L_2=L\otimes_{\mathbf{Z}}\mathbf{Z}_2=L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$. For such a choice of $L$, the group scheme $G_{\mathbf{Z}_{(2)}}$ is reductive and $G_{\mathbf{Z}_2}$ is a split $\mathbf{GSO}_{2n}$ group scheme; therefore the Main Theorem applies.

**1.4.2. On the new ideas and contents.** The deformation theories of Subsubsection 1.2.3 do not suffice to show that $\mathcal{N}^{\mathrm{n}}_{W(k)}$ is formally smooth over $W(k)$ at points above $y\in\mathcal{N}_{W(k)}(k)$ (see Subsection 1.1 and Subsubsection 4.1.1). This explains why the proof of the Main Theorem uses also some of the techniques of [14] and [38], the mentioned result of Noot, and the following three new ideas:

**(i)** We get versions of the group theoretical results [25, Lemma 7.2 and Cor. 7.3] that pertain to the case (D) in mixed characteristic $(0,2)$ (see Subsections 3.1, 3.3, 3.4, and 5.2).

**(ii)** We use the crystalline Dieudonné theory of [4] and [5] in order to prove Theorem 5.1 that surpasses (in the geometric context of the Main Theorem) the problem (i) of Subsection 1.1 and that (together with the idea (i)) is the very essence of the proof of Theorem 1.4 (a).

**(iii)** We use a modulo 2 version of Faltings deformation theory [14, Section 7] as a key ingredient in the proofs of Theorem 1.4 (b) (see Subsections 6.3 to 6.7) and of the fact that the ordinary locus of $\mathcal{N}_{k(v)}$ is Zariski dense in $\mathcal{N}_{k(v)}$ (see Proposition 7.3). Subsections 6.3 to 6.7 surpass (in the geometric context of the Main Theorem) the modulo 2 version of the problems (ii) and (iii) of Subsection 1.1.

Ideas (i) and (ii) are specific to the PEL context of Subsection 1.3. But the idea (iii) can be easily adapted to all characteristics and to all Shimura varieties of Hodge type.

In Sections 2 and 3 we present tools from the crystalline theory and from the theories of group schemes and of involutions of matrix algebras over commutative $\mathbf{Z}$-algebras (respectively). In Sections 5 to 7 we prove Theorem 1.4 (a), (b), and (c) (respectively). In Section 4 we list crystalline notations and elementary properties that are needed in Sections 5 to 7.

**1.5. On Part II.** The hypothesis of Main Theorem that $G_{\mathbf{Z}_2}$ is a split $\mathbf{GSO}_{2n}$ group scheme is inserted to ease notations and to be able to apply the mentioned result of Noot. Part II of this paper will prove the Main Theorem without this hypothesis. The proofs of Theorem 1.4 (a) and (b) can be easily adapted to the general case. The main idea of Part II will be to use relative PEL situations (similar but different from the ones of [38, Subsubsection 4.3.16 and Section 6]) to get that for proving the identity $\mathcal{N}^{\mathrm{s}} = \mathcal{N}^{\mathrm{n}}$ we can assume that $G_{\mathbf{Z}_2}$ is a split $\mathbf{GSO}_{2n}$ group scheme.

## 2. Crystalline preliminaries

Let $B(k)$ be the field of fractions of $W(k)$. We denote also by $\sigma$ the Frobenius automorphism of $B(k)$. If $Z$ is a $W(k)$-scheme annihilated by some power of 2, we use Berthelot's crystalline site $CRIS(Z/\mathrm{Spec}(W(k)))$ of [3, Ch. III, Section 4]. See [4] and [5] for the crystalline Dieudonné functor $\mathbf{D}$. If $\mathcal{E}$ (resp. $E$) is a 2-divisible group (resp. a finite, flat group scheme annihilated by 2) over some $W(k)$-scheme, let $\mathcal{E}^{\mathrm{t}}$ (resp. $E^{\mathrm{t}}$) be its Cartier dual. For an affine morphism $\mathrm{Spec}(S_1) \to \mathrm{Spec}(S)$ and for $Z$ (or $Z_S$ or $Z_*$) an $S$-scheme, let $Z_{S_1}$ (or $Z_{S_1}$ or $Z_{*S_1}$) be $Z \times_S S_1$. A bilinear form $\lambda_M$ on a free $S$-module $M$ of finite rank is called perfect if it induces an $S$-linear isomorphism $M \tilde{\to} \mathrm{Hom}(M, S)$; if $\lambda_M$ is alternating, $(M, \lambda_M)$ is called a symplectic space over $S$. Let $\boldsymbol{\mu}_{2S}$ be the 2-torsion subgroup scheme of the rank 1 split torus $\mathbf{G}_{mS}$ over $S$. Let $\mathbf{G}_m(S)$ be the group of invertible elements of $S$. If $S$ is an $\mathbf{F}_2$-algebra, let $\Phi_S$ be the Frobenius endomorphism of either $S$ or $\mathrm{Spec}(S)$ (thus $\Phi_k = \sigma$), let $M^{(2)} := M \otimes_{S\ \Phi_S} S$, let $Z^{(2)} := Z \times_{S\ \Phi_S} S$, and let $\boldsymbol{\alpha}_{2S}$ be the finite, flat, group scheme over $S$ of global functions of square 0.

**2.1. Ramified data.** Let $V$ be a discrete valuation ring that is a finite extension of $W(k)$. Let $K$ be the field of fractions of $V$. Let $e := [V : W(k)]$. Let $R := W(k)[[t]]$, with $t$ as an independent variable. We fix a uniformizer $\pi$ of $V$. Let $f_e \in R$ be the Eisenstein polynomial of degree $e$ that has $\pi$ as a root. Let $R_e$ be the local $W(k)$-subalgebra of $B(k)[[t]]$ formed by formal power series $\sum_{i=0}^{\infty} a_i t^i$ with the properties that for all $i \in \mathbf{N} \cup \{0\}$ we have $b_i := a_i[\frac{i}{e}]! \in W(k)$ and that the sequence $(b_i)_{i\in\mathbf{N}}$ of elements of $W(k)$ converges to 0. Let $I_e(1) := \{\sum_{i=0}^{\infty} a_i t^i \in R_e | a_0 = 0\}$. Let $\Phi_{R_e}$ be the Frobenius lift of $R_e$ that is compatible with $\sigma$ and that takes $t$ to $t^2$. For $m \in \mathbf{N}$ let

$$U_m := k[t]/(t^m) = k[[t]]/(t^m).$$

Let $S_e$ be the $W(k)$-subalgebra of $B(k)[t]$ generated by $W(k)[t]$ and by the $\frac{f_e^i}{i!}$'s with $i \in \mathbf{N} \cup \{0\}$. As $f_e$ is an Eisenstein polynomial, $S_e$ is also $W(k)$-generated by $W(k)[t]$ and by the $\frac{t^{ei}}{i!}$'s with $i \in \mathbf{N} \cup \{0\}$. Therefore the $W(k)$-algebra $S_e$ depends (as its

notation suggests) only on $e$. The 2-adic completion of $S_e$ is $R_e$. Thus we have a $W(k)$-epimorphism $R_e \twoheadrightarrow V$ that takes $t$ to $\pi$ and whose kernel has a natural divided power structure. We identify canonically $R_e/I_e(1) = W(k)$, $V = R/(f_e)$, and $V/2V = U_e$.

**2.2. On finite, flat group schemes.** Let $E_0$ be a finite, flat group scheme annihilated by 2 over a commutative $W(k)$-algebra $S$; this implies that $E_0$ is commutative. Let $\Delta_{E_0} : E_0 \hookrightarrow E_0 \times_S E_0$ be the diagonal monomorphism of $S$-schemes.

**2.2.1. Definition.** By a *principal quasi-polarization* of $E_0$ we mean an isomorphism $\lambda_{E_0} : E_0 \tilde{\to} E_0^{\mathrm{t}}$ over $S$ such that the composite $co_{E_0} : E_0 \to \boldsymbol{\mu}_{2S}$ of $(1_{E_0}, \lambda_{E_0}) \circ \Delta_{E_0}$ with the coupling morphism $cu_{E_0} : E_0 \times_S E_0^{\mathrm{t}} \to \boldsymbol{\mu}_{2S}$ factors through the identity section of $\boldsymbol{\mu}_{2S}$.

Let $r \in \mathbf{N}$. Until Section 3 we will take $S = V$, $E_0$ to be of order $2^{2r}$, and $\lambda_{E_0}$ to be a principal quasi-polarization. Thus the perfect bilinear form on $E_{0K}$ induced by $\lambda_{E_0}$ is alternating. Let $(N_0, \phi_0, \upsilon_0, \nabla_0, b_{N_0})$ be the evaluation of $\mathbf{D}(E_0, \lambda_{E_0})$ (equivalently of $\mathbf{D}((E_0, \lambda_{E_0})_{U_e}))$ at the trivial thickening of $\mathrm{Spec}(U_e)$. Thus $N_0$ is a free $U_e$-module of rank $2r$, the maps $\phi_0 : N_0^{(2)} \to N_0$ and $\upsilon_0 : N_0 \to N_0^{(2)}$ are $U_e$-linear, $\nabla_0$ is a connection on $N_0$, and $b_{N_0}$ is a perfect bilinear form on $N_0$. We have $\phi_0 \circ \upsilon_0 = 0$ and $\upsilon_0 \circ \phi_0 = 0$. As $\lambda_{E_0}$ is a principal quasi-polarization, the bilinear morphism $cu_{E_0} \circ (1_{E_0}, \lambda_{E_0})$ over $S$ is symmetric. Thus $b_{N_0}$ is a symmetric form on $N_0$. We expect that (at least) under mild conditions the form $b_{N_0}$ on $N_0$ is in fact alternating. Here is an example over $k$ that implicitly points out that some conditions might be indeed in order.

**2.2.2. Example.** Let $E_1$ be the 2-torsion group scheme of the unique supersingular 2-divisible group over $k$ of height 2. The evaluation $(N_1, \phi_1, \upsilon_1)$ of $\mathbf{D}(E_1)$ at the trivial thickening of $\mathrm{Spec}(k)$ is a 2-dimensional $k$-vector space spanned by elements $e_1$ and $f_1$ that satisfy $\phi_1(e_1) = \upsilon_1(e_1) = f_1$ and $\phi_1(f_1) = \upsilon_1(f_1) = 0$. Let $b_{N_1}$ be the perfect, non-alternating symmetric bilinear form on $N_1$ given by the rules: $b_{N_1}(f_1, e_1) = b_{N_1}(e_1, e_1) = b_{N_1}(e_1, f_1) = 1$ and $b_{N_1}(f_1, f_1) = 0$. For $u$, $x \in N_1$ we have $b_{N_1}(u, \phi_1(x)) = b_{N_1}(\phi_1(x), u) = b_{N_1}(x, \upsilon_1(u))^2 = b_{N_1}(\upsilon_1(u), x)$. It is well known that the perfect bilinear form $b_{N_1}$ gives birth to an isomorphism $\lambda_{E_1} : E_1 \tilde{\to} E_1^{\mathrm{t}}$ over $k$. As we are in characteristic 2, the morphism $co_{E_1} : E_1 \to \boldsymbol{\mu}_{2k}$ over $k$ is a homomorphism over $k$. But as $E_1$ has no toric part, $co_{E_1}$ is trivial. Thus $\lambda_{E_1}$ is a principal quasi-polarization.

**2.2.3. Lemma.** *If $V = W(k)$, then the perfect form $b_{N_0}$ on $N_0$ is alternating.*

*Proof:* Let $\mathbf{D}(E_0) = (N_0, F_0, \phi_0, \phi_1, \nabla_0)$ be the Dieudonné module of $E_0$ as used in [39, Construction 2.2]. Thus $F_0 = \mathrm{Ker}(\phi_0)$ and $\phi_1 : F_0 \to N_0$ is a $\sigma$-linear map such that we have $N_0 = \phi_0(N_0) + \phi_1(F_0)$. Let $x \in N_0$ and $u \in F_0$. We have $b_{N_0}(\phi_0(x), \phi_0(x)) = 2b_{N_0}(x, x)^2 = 0$ and $b_{N_0}(\phi_1(u), \phi_1(u)) = 2b_{N_0}(u, u)^2 = 0$. Let $v \in N_0$. We choose $x$ and $u$ such that we have an identity $v = \phi_0(x) + \phi_1(u)$. As $b_{N_0}$ is symmetric, we compute that $b_{N_0}(v, v) = b_{N_0}(\phi_0(x), \phi_0(x)) + b_{N_0}(\phi_1(u), \phi_1(u)) = 0 + 0 = 0$. Thus $b_{N_0}$ is alternating.$\square$

If $V \neq W(k)$, then the ring $V/4V$ has no Frobenius lift. Thus the proof of Lemma 2.2.3 can not be adapted if $V \neq W(k)$. Based on this, for $V \neq W(k)$ we will study pairs

of the form $(N_0, b_{N_0})$ only in some particular cases related naturally to the geometric context of Theorem 1.4 (see Theorem 5.1 below).

## 3. Group schemes and involutions

Let $n \in \mathbf{N}$. Let $\mathrm{Spec}(S)$ be an affine scheme. We recall that a reductive group scheme $\mathcal{R}$ over $S$ is a smooth, affine group scheme over $S$ whose fibres are connected and have trivial unipotent radicals. Let $\mathcal{R}^{\mathrm{ad}}$ and $\mathcal{R}^{\mathrm{der}}$ be the adjoint and the derived (respectively) group schemes of $\mathcal{R}$, cf. [12, Vol. III, Exp. XXII, Def. 4.3.6 and Thm. 6.2.1]. Let $\mathrm{Lie}(\mathcal{U})$ be the Lie algebra of a smooth, closed subgroup scheme $\mathcal{U}$ of $\mathcal{R}$. If $M$ is a free $S$-module of finite rank, let $M^* := \mathrm{Hom}(M, S)$, let $\mathbf{GL}_M$ be the reductive group scheme over $S$ of linear automorphisms of $M$, and let $\mathcal{T}(M) := \oplus_{s,m \in \mathbf{N}\cup\{0\}} M^{\otimes s} \otimes_S M^{*\otimes m}$. Each $S$-linear isomorphism $i_S : M \tilde{\to} \tilde{M}$ of free $S$-modules of finite rank, extends naturally to an $S$-linear isomorphism (denoted also by) $i_S : \mathcal{T}(M) \tilde{\to} \mathcal{T}(\tilde{M})$ and thus we will speak about $i_S$ taking some tensor of $\mathcal{T}(M)$ to some tensor of $\mathcal{T}(\tilde{M})$. We identify $\mathrm{End}(M) = M \otimes_S M^*$. If $(M, \lambda_M)$ is a symplectic space over $S$, let $\mathbf{Sp}(M, \lambda_M) := \mathbf{GSp}(M, \lambda_M)^{\mathrm{der}}$. We often use the same notation for two elements of some modules (like involutions, endomorphisms, bilinear forms, etc.) that are obtained one from another via extensions of scalars and restrictions. The reductive group schemes $\mathbf{GL}_{nS}$, $\mathbf{Sp}_{2nS}$, etc., are over $S$.

Let $\mathbf{SO}^+_{2nS}$, $\mathbf{GSO}^+_{2nS}$, and $\mathbf{O}^+_{2nS}$ be the split $\mathbf{SO}_{2n}$, $\mathbf{GSO}_{2n}$, and $\mathbf{O}_{2n}$ (respectively) group schemes over $S$. We recall that $\mathbf{GSO}^+_{2nS}$ is the quotient of $\mathbf{SO}^+_{2nS} \times_S \mathbf{G}_{mS}$ by a $\boldsymbol{\mu}_{2S}$ subgroup scheme that is embedded diagonally.

In Subsections 3.1 and 3.2 we present some general facts on $\mathbf{SO}^+_{2n}$ group schemes in mixed characteristic $(0, 2)$; see [7, plates I and IV] for the weights used. In Subsections 3.3 and 3.4 we include complements on involutions and on non-alternating symmetric bilinear forms. In Subsection 3.5 we review some standard properties of the Shimura pair $(G, \mathcal{X})$ we introduced in Subsection 1.3.

**3.1. The $\mathcal{D}_n$ group scheme.** We consider the quadratic form

$$\mathfrak{Q}_n(x) := x_1x_2 + \cdots + x_{2n-1}x_{2n} \quad \text{defined for} \quad x = (x_1, \ldots, x_{2n}) \in \mathcal{L}_n := \mathbf{Z}_{(2)}^{2n}.$$

For $\alpha \in \mathbf{Z}_{(2)}$ and $x \in \mathcal{L}_n$ we have $\mathfrak{Q}_n(\alpha x) = \alpha^2 \mathfrak{Q}_n(x)$. Let $\tilde{\mathcal{D}}_n$ be the closed subgroup scheme of $\mathbf{GL}_{\mathcal{L}_n}$ that fixes $\mathfrak{Q}_n$. Let $\mathcal{D}_n$ be the Zariski closure of the identity component of $\tilde{\mathcal{D}}_{n\mathbf{Q}}$ in $\tilde{\mathcal{D}}_n$. The fibres of $\tilde{\mathcal{D}}_n$ are smooth and have identity components that are split reductive groups (see [6, Subsection 23.6] for the fibre of $\tilde{\mathcal{D}}_n$ over $\mathbf{F}_2$). We get that $\mathcal{D}_n$ is a smooth, affine group scheme over $\mathbf{Z}_{(2)}$ whose generic fibre is connected and whose special fibre $\mathcal{D}_{n\mathbf{F}_2}$ has an identity component which is a split reductive group. Thus $\mathcal{D}_{n\mathbf{F}_2}$ is connected, cf. [37, Subsubsection 3.8.1] applied to $\mathcal{D}_{n\mathbf{Z}_2}$. Therefore $\mathcal{D}_n$ is a reductive group scheme; it has a maximal split torus of rank $n$ and thus it is split. We conclude that $\mathcal{D}_n$ is isomorphic to $\mathbf{SO}^+_{2n\mathbf{Z}_{(2)}}$ (cf. the uniqueness of a split reductive group scheme associated to a root datum; see [12, Vol. III, Exp. XXIII, Cor. 5.1]). We get that $\mathcal{D}_1$ is isomorphic to $\mathbf{G}_{m\mathbf{Z}_{(2)}}$, that $\mathcal{D}_2$ is the quotient of a product of two $\mathbf{SL}_{2\mathbf{Z}_{(2)}}$ group schemes

by a $\boldsymbol{\mu}_{2\mathbf{Z}_{(2)}}$ subgroup scheme which is embedded diagonally, and that for $n \geq 3$ the group scheme $\mathcal{D}_n$ is semisimple and $\mathcal{D}_n^{\mathrm{ad}}$ is a split, absolutely simple, adjoint group scheme of $D_n$ Lie type.

Next we study the rank $2n$ faithful representation

$$\rho_n : \mathcal{D}_n \hookrightarrow \mathbf{GL}_{\mathcal{L}_n}.$$

If $n \geq 4$, then $\rho_n$ is associated to the minimal weight $\varpi_1$. But $\rho_3$ is associated to the weight $\varpi_2$ of the $A_3$ Lie type and $\rho_2$ is the tensor product of the standard rank 2 representations of the mentioned two $\mathbf{SL}_{2\mathbf{Z}_{(2)}}$ group schemes. Thus $\rho_n$ is isomorphic to its dual and, up to a $\mathbf{G}_m(\mathbf{Z}_{(2)})$-multiple, there exists a unique perfect, symmetric bilinear form $\mathfrak{B}_n$ on $\mathcal{L}_n$ fixed by $\mathcal{D}_n$ (the case $n = 1$ is trivial). In fact we can take $\mathfrak{B}_n$ such that we have

$$\mathfrak{B}_n(u, x) := \mathfrak{Q}_n(u + x) - \mathfrak{Q}_n(u) - \mathfrak{Q}_n(x) \quad \forall u, x \in \mathcal{L}_n.$$

We can recover $\mathfrak{Q}_n$ from $\mathfrak{B}_n$ via the formula $\mathfrak{Q}_n(x) = \frac{\mathfrak{B}_n(x,x)}{2}$, where $x \in \mathcal{L}_n$. This formula makes sense as 2 is a non-zero divisor in $\mathbf{Z}_{(2)}$. We emphasize that we can not recover in a canonical way the reduction modulo $2\mathbf{Z}_{(2)}$ of $\mathfrak{Q}_n$ from the reduction modulo $2\mathbf{Z}_{(2)}$ of $\mathfrak{B}_n$. If $n \geq 2$, then (due to reasons of dimensions) $\mathcal{D}_{n\mathbf{F}}$ has no faithful representation of dimension at most $n$ and therefore the special fibre $\rho_{n\mathbf{F}_2}$ of $\rho_n$ is an absolutely irreducible representation. Let $J(2n)$ be the matrix of $\mathfrak{B}_n$ with respect to the standard $\mathbf{Z}_{(2)}$-basis for $\mathcal{L}_n$; it is formed by $n$ diagonal blocks that are $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore $\mathfrak{B}_n$ modulo $2\mathbf{Z}_{(2)}$ is alternating.

As $\tilde{\mathcal{D}}_{n\mathbf{F}_2}$ is smooth, we easily get that $\mathcal{D}_n$ is the identity component of $\tilde{\mathcal{D}}_n$. We check that $\tilde{\mathcal{D}}_n$ is isomorphic to $\mathbf{O}^+_{2n\mathbf{Z}_{(2)}}$ i.e., we have a non-trivial short exact sequence

$$0 \to \mathcal{D}_n \to \tilde{\mathcal{D}}_n \to \mathbf{Z}/2\mathbf{Z}_{\mathbf{Z}_{(2)}} \to 0 \tag{2}$$

that splits. We can assume that $n \geq 2$ (as the case $n = 1$ is easy). As $\rho_{n\mathbf{F}_2}$ is absolutely irreducible for $n \geq 2$, the centralizer of $\mathcal{D}_{n\mathbf{F}_2}$ in $\tilde{\mathcal{D}}_{n\mathbf{F}_2}$ is a $\boldsymbol{\mu}_{2\mathbf{F}_2}$ group scheme and thus connected. Thus the group of connected components of $\tilde{\mathcal{D}}_{n\mathbf{F}_2}$ acts on $\mathcal{D}_{n\mathbf{F}_2}$ via outer automorphisms that are as well automorphisms of $\rho_{n\mathbf{F}_2}$. This implies that $\tilde{\mathcal{D}}_{n\mathbf{F}_2}$ has at most two connected components (even if $2 \leq n \leq 4$). But it is well known that $\tilde{\mathcal{D}}_{n\mathbf{Q}}$ has two connected components and thus the open closed subscheme $\tilde{\mathcal{D}}_n \setminus \mathcal{D}_n$ of $\tilde{\mathcal{D}}_n$ is flat and has a connected special fibre. We conclude that $\tilde{\mathcal{D}}_n$ is a flat group scheme over $\mathbf{Z}_{(2)}$ and that the short exact sequence (2) exists. The $2n \times 2n$ block matrix $\left(\begin{smallmatrix} J(2) & O \\ O & I_{2n-2} \end{smallmatrix}\right)$ is an element of $(\tilde{\mathcal{D}}_n \setminus \mathcal{D}_n)(\mathbf{Z}_{(2)})$ of order 2 and thus the short exact sequence (2) splits (here $I_{2n-2}$ is the $2n-2 \times 2n-2$ identity matrix). The short exact sequence (2) is non-trivial as its generic fibre is non-trivial. Next we recall an obvious Fact.

**3.1.1. Fact.** *Let $b_N$ be a symmetric bilinear form on a free module $N$ of finite rank over a commutative $\mathbf{F}_2$-algebra $S$. We have:*

**(a)** *The quadratic map $q_N : N \to S$ that maps $x \in N$ to $b_N(x,x) \in S$ is additive and its kernel* $\mathrm{Ker}(q_N)$ *is an $S$-module. We have $N = \mathrm{Ker}(q_N)$ if and only if $b_N$ is alternating.*

**(b)** *If $S$ is a field, then* $\dim_S(N/\mathrm{Ker}(q_N)) \leq [S : \Phi_S(S)]$.

**3.1.2. Fact.** *Let $\mathcal{G}_n$ be the flat, closed subgroup scheme of $\mathbf{GL}_{\mathcal{L}_n}$ generated by $\mathcal{D}_n$ and by the center $\mathcal{Z}_n$ of $\mathbf{GL}_{\mathcal{L}_n}$. Then $\mathcal{G}_n$ is a reductive group scheme isomorphic to $\mathbf{GSO}_{2n}{}^{+}_{\mathbf{Z}_{(2)}}$.*

*Proof:* The kernel of the product homomorphism $\Pi_n : \mathcal{D}_n \times_{\mathbf{Z}_{(2)}} \mathcal{Z}_n \to \mathbf{GL}_{\mathcal{L}_n}$ is isomorphic to $\boldsymbol{\mu}_{2\mathbf{Z}_{(2)}}$. The quotient group scheme $\mathcal{D}_n \times_{\mathbf{Z}_{(2)}} \mathcal{Z}_n/\mathrm{Ker}(\Pi_n)$ is reductive, cf. [12, Vol. III, Exp. XXII, Prop. 4.3.1]. The resulting homomorphism $\mathcal{D}_n \times_{\mathbf{Z}_{(2)}} \mathcal{Z}_n/\mathrm{Ker}(\Pi_n) \to \mathbf{GL}_{\mathcal{L}_n}$ has fibres that are closed embeddings and thus (cf. [12, Vol. II, Exp. XVI, Cor. 1.5 a)]) it is a closed embedding. Thus we can identify $\mathcal{G}_n$ with $\mathcal{D}_n \times_{\mathbf{Z}_{(2)}} \mathcal{Z}_n/\mathrm{Ker}(\Pi_n)$. Therefore $\mathcal{G}_n$ is a reductive group scheme isomorphic to $\mathbf{GSO}_{2n}{}^{+}_{\mathbf{Z}_{(2)}}$. $\square$

**3.2. Lemma.** *Let $S$ be a commutative, faithfully flat $\mathbf{Z}_{(2)}$-algebra. Let $\mathcal{L}_S$ be a free $S$-submodule of $\mathcal{L}_n \otimes_{\mathbf{Z}_{(2)}} S[\frac{1}{2}]$ of rank $2n$ such that $\mathcal{L}_S[\frac{1}{2}] = \mathcal{L}_n \otimes_{\mathbf{Z}_{(2)}} S[\frac{1}{2}]$ and we get a perfect bilinear form $\mathfrak{B}_n : \mathcal{L}_S \otimes_S \mathcal{L}_S \to S$. Suppose that there exists a reductive, closed subgroup scheme $\mathcal{D}_n(\mathcal{L}_S)$ of $\mathbf{GL}_{\mathcal{L}_S}$ whose fibre over $S[\frac{1}{2}]$ is $\mathcal{D}_{nS[\frac{1}{2}]}$ (therefore if $S$ is reduced, then $\mathcal{D}_n(\mathcal{L}_S)$ is the Zariski closure of $\mathcal{D}_{nS[\frac{1}{2}]}$ in $\mathbf{GL}_{\mathcal{L}_S}$). Then the perfect bilinear form on $\mathcal{L}_S/2\mathcal{L}_S$ induced by $\mathfrak{B}_n$ is alternating.*

*Proof:* The statement of the Lemma is local in the flat topology of $\mathrm{Spec}(S)$. Thus we can assume that $S$ is local and $\mathcal{D}_n(\mathcal{L}_S)$ is split, cf. [12, Vol. III, Exp. XIX, Prop. 6.1]. Let $T_1$ and $T_2$ be maximal split tori of $\mathcal{D}_{nS}$ and $\mathcal{D}_n(\mathcal{L}_S)$ (respectively). Let $h \in \mathcal{D}_n^{\mathrm{ad}}(S[\frac{1}{2}])$ be such that $h(T_{1S[\frac{1}{2}]})h^{-1} = T_{2S[\frac{1}{2}]}$, cf. [12, Vol. III, Exp. XXIV, Lemma 1.5]. By replacing $\mathrm{Spec}(S)$ with a torsor of the center of $\mathcal{D}_n(\mathcal{L}_S)$ if $n \geq 2$, we can assume that there exists $g \in \mathcal{D}_n(S[\frac{1}{2}])$ that maps to $h$. By replacing $\mathcal{L}_S$ with $g^{-1}(\mathcal{L}_S)$, we can assume that $T_{1S[\frac{1}{2}]} = T_{2S[\frac{1}{2}]}$. Thus we can identify $T_1 = T_2$ (for instance, cf. [12, Vol. II, Exp. X, Cor. 7.2]).

We consider the $T_1$-invariant direct sum decompositions $\mathcal{L}_n \otimes_{\mathbf{Z}_{(2)}} S = \oplus_{i=1}^{2n} \mathcal{V}_i$ and $\mathcal{L}_S = \oplus_{i=1}^{2n} \mathcal{V}_i(\mathcal{L}_S)$ into free $S$-modules of rank 1, numbered in such a way that $T_1$ acts on $\mathcal{V}_i$ and $\mathcal{V}_i(\mathcal{L}_S)$ via the same character of $T_1$ and we have $\mathfrak{B}_n(\mathcal{V}_i, \mathcal{V}_j) = 0$ if $1 \leq i < j \leq 2n$ and $(i,j) \notin \{(1,2),(3,4),\ldots,(2n-1,2n)\}$. Let $\mathcal{W}$ be an $S$-basis for $\mathcal{L}_S$ formed by elements of $\mathcal{V}_i(\mathcal{L}_S)$'s. We have $\mathcal{V}_i(\mathcal{L}_S) = f_i\mathcal{V}_i$, where $f_i \in \mathbf{G}_m(S[\frac{1}{2}])$. Thus for all $x \in \mathcal{W}$ we have $\mathfrak{B}_n(x,x) = 0$. Based on this and on the fact that $\mathfrak{B}_n$ is symmetric, from Fact 3.1.1 (a) we get that the perfect bilinear form on $\mathcal{L}_S/2\mathcal{L}_S$ induced by $\mathfrak{B}_n$ is alternating. $\square$

**3.3. Involutions.** See [24, Ch. I] for standard terminology and properties of involutions of (finite dimensional) semisimple algebras over fields. Let $\mathrm{Spec}(S)$ be a connected, reduced, affine scheme. Let $\mathcal{F} = \oplus_{i=1}^{s} \mathcal{F}_i$ be a product of matrix $S$-algebras. Let $I_{\mathcal{F}}$ be an involution of $\mathcal{F}$ that fixes $S$. Thus $I_{\mathcal{F}}$ is an $S$-linear automorphism of $\mathcal{F}$ of order 2 such that for all $x$, $u \in \mathcal{F}$ we have $I_{\mathcal{F}}(xu) = I_{\mathcal{F}}(u)I_{\mathcal{F}}(x)$. Let $\eta$ be the permutation of $\{1,\ldots,s\}$ such that we have $I_{\mathcal{F}}(\mathcal{F}_i) = \mathcal{F}_{\eta(i)}$, for all $i \in \{1,\ldots,s\}$. The order of $\eta$ is either 1 or 2.

Let $C_0$ be a subset of $\{1, \ldots, s\}$ whose elements are permuted transitively by $\eta$. Let $\mathcal{F}_0 := \oplus_{i \in C_0} \mathcal{F}_i$. Let $I_0$ be the restriction of $\mathcal{I}_{\mathcal{F}}$ to $\mathcal{F}_0$. We refer to $(\mathcal{F}_0, I_0)$ as a simple factor of $(\mathcal{F}, I_{\mathcal{F}})$. The simple factor $(\mathcal{F}_0, I_0)$ is said to be:

– of second type, if $C_0$ has two elements;
– of first type, if $C_0$ has only one element and $I_0 \neq 1_{\mathcal{F}_0}$;
– trivial, if $C_0$ has only one element and $I_0 = 1_{\mathcal{F}_0}$.

**3.3.1. Lemma.** *Suppose that $S$ is also local and $(\mathcal{F}_0, I_0)$ is of first type (thus $\mathcal{F}_0$ is $\mathcal{F}_i$ for some $i \in \{1, \ldots, s\}$). We identify $\mathcal{F}_0 = \mathrm{End}(M_0)$, where $M_0$ is a free $S$-module of finite rank. We have:*

**(a)** *There exists a perfect bilinear form $b_0$ on $M_0$ such that the involution of $\mathcal{F}_0$ is defined by $b_0$ i.e., for all $u$, $v \in M_0$ and all $x \in \mathcal{F}_0$ we have $b_0(x(u), v) = b_0(u, I_0(x)(v))$.*

**(b)** *The form $b_0$ (of (a)) is uniquely determined up to a $\mathbf{G}_m(S)$-multiple.*

**(c)** *If $S$ is also integral, then $b_0$ is either alternating or non-alternating symmetric.*

*Proof:* Let $\mathrm{Aut}(\mathbf{GL}_{M_0})$ be the group scheme of automorphisms of $\mathbf{GL}_{M_0}$. To prove (a) we follow [24, Ch. I]. We identify the opposite $S$-algebra of $\mathcal{F}_0$ with $\mathrm{End}(M_0^*)$. Therefore $I_0$ defines naturally an $S$-isomorphism $I_0 : \mathrm{End}(M_0) \tilde{\to} \mathrm{End}(M_0^*)$. We claim that each such $S$-isomorphism is defined naturally by an $S$-linear isomorphism $c_0 : M_0 \tilde{\to} M_0^*$. To check this claim, we fix an $S$-isomorphism $I_0^1 : \mathrm{End}(M_0^*) \tilde{\to} \mathrm{End}(M_0)$ defined by an $S$-linear isomorphism $c_1 : M_0^* \tilde{\to} M_0$ and it suffices to show that the $S$-automorphism $a_0 := I_0^1 \circ I_0$ of $\mathrm{End}(M_0)$ is defined (via inner conjugation) by an element $d_0 \in \mathbf{GL}_{M_0}(S)$. But $a_0$ defines naturally an $S$-valued automorphism $\tilde{a}_0 \in \mathrm{Aut}(\mathbf{GL}_{M_0})(S)$ of $\mathbf{GL}_{M_0}$.

Let $\mathbf{PGL}_{M_0} := \mathbf{GL}_{M_0}^{\mathrm{ad}}$. We have a short exact sequence $0 \to \mathbf{PGL}_{M_0} \to \mathrm{Aut}(\mathbf{GL}_{M_0}) \to E_0 \to 0$, where the group scheme $E_0$ over $S$ is either trivial or $\mathbf{Z}/2\mathbf{Z}_S$ (cf. [12, Vol. III, Exp. XXIV, Thm. 1.3]). We check that $\tilde{a}_0 \in \mathbf{PGL}_{M_0}(S)$. We need to show that the image $\tilde{a}_2$ of $\tilde{a}_0$ in $E_0(S)$ is the identity element. As $S$ is reduced, to check this last thing we can assume that $S$ is a field and therefore the fact that $\tilde{a}_2$ is the identity element is implied by the Skolem–Noether theorem. Thus $\tilde{a}_0 \in \mathbf{PGL}_{M_0}(S)$.

As $S$ is local, all torsors of $\mathbf{G}_{mS}$ are trivial. Thus there exists $d_0 \in \mathbf{GL}_{M_0}(S)$ that maps to $\tilde{a}_0$. Therefore $c_0 := c_1^{-1} \circ d_0$ exists. Let $b_0$ be the perfect bilinear form on $M_0$ defined naturally by $c_0$. As $S$ is reduced, as in the case of fields one checks that for all $u$, $v \in M_0$ and all $x \in \mathcal{F}_0$ we have $b_0(x(u), v) = b_0(u, I_0(x)(v))$. This proves (a).

We check (b). Both $b_0$ and $c_0$ are uniquely determined by $d_0$. As $a_0$ and $\tilde{a}_0$ are uniquely determined by $I_0$, we get that $d_0$ is uniquely determined by $I_0$ up to a scalar multiplication with an element of $\mathbf{G}_m(S)$. From this (b) follows. For (c) we can assume that $S$ is a field and this case is well known (for instance, see [24, Ch. I, 2.1]). $\square$

**3.3.2. Definition.** Suppose that $S$ is local and integral. Let $b_0$ be as in Lemma 3.3.1 (a). Let $J$ be an ideal of $S$. If the reduction of $b_0$ modulo $J$ is alternating (resp. is non-alternating symmetric), then we say that the reduction of the simple factor $(\mathcal{F}_0, I_0)$ modulo $J$ is of alternating (resp. of orthogonal) first type.

We have the following converse form of Lemma 3.2.

**3.4. Proposition.** *Suppose that the commutative $\mathbf{Z}_{(2)}$-algebra $S$ is local, noetherian, 2-adically complete, and strictly henselian. Let $b_M$ be a perfect, non-alternating symmetric bilinear form on a free $S$-module $M$ of finite rank $2n$ with the property that $b_M$ modulo $2S$ is alternating. We have:*

**(a)** *There exists an $S$-basis $\mathcal{W}$ for $M$ with respect to which the matrix of $b_M$ is $J(2n)$.*

**(b)** *Suppose that $S$ is as well a faithfully flat $\mathbf{Z}_{(2)}$-algebra. Let $q_M : M \to S$ be the quadratic form defined by the rule $q_M(x) := \frac{b_M(x,x)}{2}$, where $x \in M$. Let $\mathbf{O}(M, q_M)$ be the closed subgroup scheme of $\mathbf{GL}_M$ that fixes $q_M$. Let $\mathbf{SO}(M, q_M)$ be the identity component of $\mathbf{O}(M, q_M)$. Let $\mathbf{GSO}(M, q_M)$ be the closed subgroup scheme of $\mathbf{GL}_M$ generated by $\mathbf{SO}(M, q_M)$ and by the center of $\mathbf{GL}_M$. Then $\mathbf{O}(M, q_M)$ (resp. $\mathbf{SO}(M, q_M)$ or $\mathbf{GSO}(M, q_M)$) is isomorphic to $\tilde{\mathcal{D}}_{nS}$ (resp. $\mathcal{D}_{nS}$ or $\mathcal{G}_{nS}$). If $S$ is also reduced and if $K_S$ denotes its ring of fractions, then $\mathbf{SO}(M, q_M)$ is the Zariski closure of $\mathbf{SO}(M, q_M)_{K_S}$ in $\mathbf{GL}_M$.*

*Proof:* We prove (a). Our hypotheses imply that $S/2S$ is local, noetherian, and strictly henselian. As $b_M$ modulo $2S$ is alternating, there exists an $S/2S$-basis for $M/2M$ with respect to which the matrix of $b_M$ modulo $2S$ is $J(2n)$. By induction on $q \in \mathbf{N}$ we show that there exists an $S/2^qS$-basis for $M/2^qM$ with respect to which the matrix of $b_M$ modulo $2^qS$ is $J(2n)$. The passage from $q$ to $q+1$ goes as follows. We denote also by $b_M$ different evaluations of its reduction modulo $2^qS$ (resp. $2^{q+1}S$) at elements of $M/2^qM$ (resp. of $M/2^{q+1}M$). Let $\mathcal{W}_q = \{u_{1,q}, v_{1,q}, \ldots, u_{n,q}, v_{n,q}\}$ be an $S/2^qS$-basis for $M/2^qM$ such that we have: (i) $b_M(u_{i,q}, v_{i,q}) = 1$ and $b_M(u_{i,q}, u_{j,q}) = b_M(v_{i,q}, v_{j,q}) = 0$ for all $i$, $j \in \{1, \ldots, n\}$ and (ii) $b_M(u_{i,q}, v_{j,q}) = 0$ for all $i$, $j \in \{1, \ldots, n\}$ with $i \neq j$.

Let $\mathcal{W}_{q+1} = \{u_{1,q+1}, v_{1,q+1}, \ldots, u_{n,q+1}, v_{n,q+1}\}$ be an $S/2^{q+1}S$-basis for $M/2^{q+1}M$ which modulo $2^{q-1}S/2^{q+1}S$ is $\mathcal{W}_q$ modulo $2^{q-1}S/2^qS$, which modulo $2^qS/2^{q+1}S$ is an $S/2^qS$-basis for $M/2^qM$ with respect to which the matrix of $b_M$ modulo $2^qS$ is $J(2n)$, and which is such that for all $i$, $j \in \{1, \ldots, n\}$ with $i \neq j$ we have $b_M(u_{i,q+1}), v_{j,q+1}) = b_M(u_{i,q+1}, u_{j,q+1}) = b_M(v_{i,q+1}, v_{j,q+1}) = 0$.

Let $u$, $v \in M/2^{q+1}M$ be such that $b_M(u, u) = 2^q a$, $b_M(v, v) = 2^q c$, and $b_M(u, v) = 1 + 2^q e$, where $a$, $c$, $e \in S/2^{q+1}S$. By replacing $v$ with $(1 + 2^q e)v$, we can assume that $e = 0$. We show that there exists $x \in S/2^{q+1}S$ such that

$$b_M(u + 2^{q-1}xv, u + 2^{q-1}xv) = 2^q(a + x + 2^{2q-2}x^2c)$$

is 0. If $q \geq 2$, then we can take $x = -a$. If $q = 1$, then as $x$ we can take any element of $S/2^{q+1}S$ that modulo $2S/2^{q+1}S$ is a solution of the equation in $t$

$$\bar{a} + t + \bar{c}t^2 = 0, \tag{3}$$

where $\bar{a}$ and $\bar{c} \in S/2S$ are the reductions modulo $2S/2^{q+1}S$ of $a$ and $c$ (respectively). The $S/2S$-scheme $\mathrm{Spec}(S/2S[t]/(\bar{a}+t+\bar{c}t^2))$ is étale and has points over the maximal point of $\mathrm{Spec}(S/2S)$. As $S/2S$ is strictly henselian, the equation (3) has solutions in $S/2S$. Thus $x$ exists even if $q = 1$. Therefore by replacing $(u, v)$ with $(u + 2^{q-1}xv, (1 + 2^{2q-1}xc)v)$,

we can also assume that $a=0$. Repeating the arguments, by replacing $(u,v)$ with $(u, v+2^{q-1}u'u)$, where $u' \in S/2^{q+1}S$, we can also assume that $c=0$. Thus $b_M(u,u) = b_M(v,v) = 0$ and $b_M(u,v)=1$. Under all these replacements, the $S/2^{q+1}S$-span of $\{u,v\}$ does not change.

Applying the previous paragraph to all pairs $(u,v)$ in $\{(u_{i,q+1}, v_{i,q+1}) | 1 \le i \le n\}$, we get that we can choose $\mathcal{W}_{q+1}$ such that the matrix of $b_M$ modulo $2^{q+1}S$ with respect to $\mathcal{W}_{q+1}$ is $J(2n)$. This completes the induction.

As $S$ is 2-adically complete and as $\mathcal{W}_{q+1}$ modulo $2^{q-1}S/2^{q+1}S$ is $\mathcal{W}_q$ modulo $2^{q-1}S/2^qS$, there exists an $S$-basis $\mathcal{W}$ for $M$ that modulo $2^qS$ coincides with $\mathcal{W}_{q+1}$ modulo $2^qS/2^{q+1}S$, for all $q \in \mathbf{N}$. The matrix of $b_M$ with respect to $\mathcal{W}$ is $J(2n)$. Thus (a) holds.

The triple $(S, M, q_M)$ is isomorphic to the extension to $S$ of the triple $(\mathbf{Z}_{(2)}, \mathcal{L}_n, \mathfrak{Q}_n)$ of Subsection 3.1, cf. (a). Thus (b) follows from the definitions of $\mathcal{D}_n$, $\tilde{\mathcal{D}}_n$, and $\mathcal{G}_n$ (see Subsection 3.1 and Fact 3.1.2). $\square$

**3.5. A review.** In this Subsection we use the notations of Subsection 1.3 and we review some standard properties of the Shimura pair $(G, \mathcal{X})$. Let $h : \mathbf{S} \hookrightarrow G_{\mathbf{R}}$ be an arbitrary element of $\mathcal{X}$. If $W \otimes_{\mathbf{Q}} \mathbf{C} = F_h^{-1,0} \oplus F_h^{0,-1}$ is the Hodge decomposition defined by $h$, let $\mu_h : \mathbf{G}_{m\mathbf{C}} \to G_{\mathbf{C}}$ be the Hodge cocharacter that fixes $F_h^{0,-1}$ and that acts as the identity character of $\mathbf{G}_{m\mathbf{C}}$ on $F_h^{-1,0}$. The cocharacter $\mu_h$ acts on the $\mathbf{C}$-span of $\psi$ via the identity character of $\mathbf{G}_{m\mathbf{C}}$. The image through $h$ of the compact subtorus of $\mathbf{S}$ contains the center of $\mathbf{Sp}(W,\psi)$. This implies that the normal subgroup $G^0 := G \cap \mathbf{Sp}(W,\psi)$ of $G$ that fixes $\psi$ is connected and therefore it is also reductive. Let $G^0_{\mathbf{Z}_{(2)}}$ be the Zariski closure of $G^0$ in $G_{\mathbf{Z}_{(2)}}$.

**3.5.1. Some group schemes.** Let $\mathcal{B}_1$ be the centralizer of $\mathcal{B}$ in $\mathrm{End}(L_{(2)})$. Let $G_{2\mathbf{Z}_{(2)}}$ be the centralizer of $\mathcal{B}$ in $\mathbf{GL}_{L_{(2)}}$; it is the group scheme over $\mathbf{Z}_{(2)}$ of invertible elements of $\mathcal{B}_1$. Due to the property 1.3 (i), the $W(\mathbf{F})$-algebra $\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F})$ is also a product of matrix $W(\mathbf{F})$-algebras. Thus $G_{2W(\mathbf{F})}$ is a product of $\mathbf{GL}$ groups schemes over $W(\mathbf{F})$. As $\mathcal{I}(\mathcal{B}) = \mathcal{B}$ we have also $\mathcal{I}(\mathcal{B}_1) = \mathcal{B}_1$. The involution $\mathcal{I}$ of $\mathcal{B}_1$ defines an involution (denoted also by $\mathcal{I}$) of the group of points of $G_{2\mathbf{Z}_{(2)}}$ with values in each fixed $\mathbf{Z}_{(2)}$-scheme. Let $G_1^0 := G_1 \cap \mathbf{Sp}(W,\psi)$. Let $G^0_{1\mathbf{Z}_{(2)}}$ be the Zariski closure of $G_1^0$ in $\mathbf{Sp}(L_{(2)},\psi)$; it is the maximal flat, closed subgroup scheme of $G_{2\mathbf{Z}_{(2)}}$ with the property that all its valued points are fixed by the involution $\mathcal{I}$.

Let $(\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2, \mathcal{I}) = \oplus_{j\in\kappa} (\mathcal{B}_j, \mathcal{I})$ be the product decomposition of $(\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2, \mathcal{I})$ into simple factors. Each $\mathcal{B}_j$ is a two sided ideal of $\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$ that is a product of one or two simple $\mathbf{Z}_2$-algebras. It is easy to see that $(\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F}), \mathcal{I})$ has no trivial factor and that all simple factors of $(\mathcal{B}_j, \mathcal{I}) \otimes_{\mathbf{Z}_2} W(\mathbf{F})$ have the same type. We have a product decomposition $G_{2\mathbf{Z}_2} = \prod_{j\in\kappa} G_{2j}$, where $G_{2j}$ is defined by the property that $\mathrm{Lie}(G_{2j})$ is the Lie algebra associated to $\mathcal{B}_j$. We have also product decompositions $G^0_{1\mathbf{Z}_2} = \prod_{j\in\kappa} G^0_{1j}$ and $G^0_{\mathbf{Z}_2} = \prod_{j\in\kappa} G^0_j$, where $G^0_{1j} := G^0_{1\mathbf{Z}_2} \cap G_{2j}$ and $G^0_j := G^0_{\mathbf{Z}_2} \cap G_{2j}$.

The double monomorphism $G^0_{j\,W(\mathbf{F})} \hookrightarrow G^0_{1j\,W(\mathbf{F})} \hookrightarrow G_{2j\,W(\mathbf{F})}$ over $W(\mathbf{F})$ is a product of double monomorphisms over $W(\mathbf{F})$ that are of one of the following three possible forms:

– $\mathbf{GL}_{nW(\mathbf{F})} = \mathbf{GL}_{nW(\mathbf{F})} \hookrightarrow \mathbf{GL}_{nW(\mathbf{F})} \times \mathbf{GL}_{nW(\mathbf{F})}$ (with $n \geq 2$), if each simple factor of $(\mathcal{B}_j, \mathcal{I}) \otimes_{\mathbf{Z}_2} W(\mathbf{F})$ is of second type;

– $\mathbf{Sp}_{2nW(\mathbf{F})} = \mathbf{Sp}_{2nW(\mathbf{F})} \hookrightarrow \mathbf{GL}_{2nW(\mathbf{F})}$ (with $n \in \mathbf{N}$), if each simple factor of $(\mathcal{B}_j, \mathcal{I}) \otimes_{\mathbf{Z}_2} W(\mathbf{F})$ is of symplectic first type;

– $\mathbf{SO}_{2nW(\mathbf{F})} \hookrightarrow \mathbf{O}_{2nW(\mathbf{F})} \hookrightarrow \mathbf{GL}_{2nW(\mathbf{F})}$ (with $n \geq 2$), if each simple factor of $(\mathcal{B}_j, \mathcal{I}) \otimes_{\mathbf{Z}_2} W(\mathbf{F})$ is of symplectic first type. [The case $n = 1$ is excluded as $\mathbf{SO}_{2W(\mathbf{F})}$ is a torus and as $\mathcal{B}$ is the maximal $\mathbf{Z}_{(2)}$-subalgebra of $\mathrm{End}(L_{(2)})$ fixed by $G_{\mathbf{Z}_{(2)}}$.]

Thus, as we are in the case (D), each simple factor of $(\mathcal{B}_j, \mathcal{I}) \otimes_{\mathbf{Z}_2} W(\mathbf{F})$ is of orthogonal first type. Thus the quotient group scheme $G^0_{1j\,W(\mathbf{F})}/G^0_{j\,W(\mathbf{F})}$ is a product of one or more $\mathbf{Z}/2\mathbf{Z}_{W(\mathbf{F})}$ group schemes. Let $G_{2j}{}^{\mathcal{I}}_{\mathbf{F}}$ be the subgroup of $G_{2j\mathbf{F}}$ fixed by $\mathcal{I}$. The double monomorphism $G^0_j \hookrightarrow G^0_{1j\mathbf{F}} \hookrightarrow G_{2j}{}^{\mathcal{I}}_{\mathbf{F}}$ over $\mathbf{F}$ is a product of double monomorphisms of the form $\mathbf{SO}_{2n\mathbf{F}} \hookrightarrow \mathbf{O}_{2n\mathbf{F}} \hookrightarrow \mathbf{Sp}_{2n\mathbf{F}}$ over $\mathbf{F}$ (this is so as $\mathfrak{B}_n$ modulo $2\mathbf{Z}_{(2)}$ is alternating).

We get that the quotient group $G_1/G$ is a finite, non-trivial 2-torsion group. We get also that $G^0_{\mathbf{C}}$ is isomorphic to a product of $\mathbf{SO}_{2n\mathbf{C}}$ groups ($n \geq 2$) and thus that $G^{\mathrm{der}}_{\mathbf{Z}_{(2)}} = G^0_{\mathbf{Z}_{(2)}}$. From this and [36, Subsections 2.6 and 2.7] we get that $G^0_{\mathbf{R}}$ is a product of $\mathbf{SO}^*_{2n}$ groups. [We recall that $\mathbf{SO}^*_{2n}$ is the semisimple group over $\mathbf{R}$ whose $\mathbf{R}$-valued points are those elements of $\mathbf{SL}_{2n}(\mathbf{C})$ that fix both the quadratic form $z_1^2 + \cdots + z_{2n}^2$ and the skew hermitian form $-z_1\bar{z}_{n+1} + z_{n+1}\bar{z}_1 - \cdots - z_n\bar{z}_{2n} + z_{2n}\bar{z}_n$.]

The $G(\mathbf{R})$-conjugacy class $\mathcal{X}$ of $h$ is a disjoint union of connected hermitian symmetric domains (cf. [9, Cor. 1.1.17]) that (cf. the structure of $G^0_{\mathbf{R}}$) are products of isomorphic irreducible hermitian symmetric domains of D III type i.e., of the form $\mathbf{SO}^*_{2n}(\mathbf{R})/\mathbf{U}_n(\mathbf{R})$ with $n \geq 2$. The hermitian symmetric domain $\mathbf{SO}^*_{2n}(\mathbf{R})/\mathbf{U}_n(\mathbf{R})$ has complex dimension $\frac{n(n-1)}{2}$, cf. [21, Ch. X, Section 6, Table V]. Thus if $G_{\mathbf{Z}_2}$ is isomorphic to $\mathbf{GSO}_{2n}{}^{+}_{\mathbf{Z}_2}$, then each connected component of $\mathcal{X}$ is isomorphic to $\mathbf{SO}^*_{2n}(\mathbf{R})/\mathbf{U}_n(\mathbf{R})$ and therefore has dimension $\frac{n(n-1)}{2}$.

**3.5.2. Extra tensors.** For $g \in G(\mathbf{A}_f^{(2)}) \leqslant G(\mathbf{A}_f)$, let $L_g$ be the $\mathbf{Z}$-lattice of $W$ such that we have $L_g \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}} = g(L \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})$. We have $L_{(2)} = L_g \otimes_{\mathbf{Z}} \mathbf{Z}_{(2)}$. Let $(v_\alpha)_{\alpha \in \mathcal{J}}$ be a family of tensors of $\mathcal{T}(W)$ such that $G$ is the subgroup of $\mathbf{GL}_W$ that fixes $v_\alpha$ for all $\alpha \in \mathcal{J}$, cf. [10, Prop. 3.1 (c)]. We choose the set $\mathcal{J}$ such that $\mathcal{B} \subseteq \mathcal{J}$ and for $b \in \mathcal{B}$ we have $v_b = b \in \mathrm{End}(W) = W \otimes_{\mathbf{Q}} W^*$. We recall that if $h \in \mathcal{X}$ is as in the beginning of Subsection 3.5 and if

$$w = [h, g_w] \in G_{\mathbf{Z}_{(2)}}(\mathbf{Z}_{(2)}) \backslash \mathcal{X} \times G(\mathbf{A}_f^{(2)}) = (\mathrm{Sh}(G, \mathcal{X})/H_2)(\mathbf{C}) = \mathcal{N}(\mathbf{C}),$$

then the analytic space of $w^*(\mathcal{A})$ is $L_{g_w} \backslash L \otimes_{\mathbf{Z}} \mathbf{C}/F_h^{0,-1}$, $v_b$ is de Rham realization of the $\mathbf{Z}_{(2)}$-endomorphism of $w^*(\mathcal{A})$ defined by $b \in \mathcal{B}$, and the non-degenerate alternating form on $L_{(2)} = H_1(w^*(\mathcal{A}), \mathbf{Z}_{(2)})$ defined by $w^*(\Lambda)$ is a $\mathbf{G}_m(\mathbf{Z}_{(2)})$-multiple of $\psi$ (see [30, Ch. 3] and [38, Subsection 4.1]). We get also a natural identification

$$L_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2 = (H^1_{\acute{e}t}(w^*(\mathcal{A}), \mathbf{Z}_2))^*$$

under which $v_b$ and $\psi$ get identified with the 2-adic realization of the $\mathbf{Z}_{(2)}$-endomorphism of $w^*(\mathcal{A})$ defined by $b \in \mathcal{B}$ and respectively with a $\mathbf{G}_m(\mathbf{Z}_2)$-multiple of the perfect form on $(H^1_{\acute{e}t}(w^*(\mathcal{A}), \mathbf{Z}_2))^*$ defined by $w^*(\Lambda)$.

**3.5.3. Decompositions.** In this Subsubsection we assume that $G_{\mathbf{Z}_2}$ is isomorphic to $\mathbf{GSO}_{2n}{}^+_{\mathbf{Z}_2}$. From Subsubsection 3.5.1 we easily get that $n \geq 2$ and that $G_{2\mathbf{Z}_2}$ is isomorphic to $\mathbf{GL}_{2n\mathbf{Z}_2}$. Thus we can identify $\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$ with $\mathrm{End}(\mathcal{V})$, where $\mathcal{V}$ is a free $\mathbf{Z}_2$-module of rank $2n$. Let $s \in \mathbf{N} \setminus \{1\}$ be such that as $\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$-modules we can identify

$$L_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2 = \mathcal{V}^s. \tag{4}$$

Let $b_{\mathcal{V}}$ be a perfect bilinear form on $\mathcal{V}$ that defines the involution $\mathcal{I}$ of $\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$ (cf. Lemma 3.3.1 (a)). Thus $b_{\mathcal{V}}$ is unique up to a $\mathbf{G}_m(\mathbf{Z}_2)$-multiple (cf. Lemma 3.3.1 (b)), it is fixed by $G^{\mathrm{der}}_{\mathbf{Z}_2} = G^0_{\mathbf{Z}_2}$, and it is symmetric (as $(\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2, \mathcal{I})$ is of orthogonal first type).

The projection of $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$ on any factor $\mathcal{V}$ associated naturally to (4), is an element of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$; thus it can be identified with an idempotent $\mathbf{Z}_2$-endomorphism of $\mathcal{A}_{\mathcal{N}_{W(k)}}$ and therefore also with an idempotent endomorphism of the 2-divisible group of $\mathcal{A}_{\mathcal{N}_{W(k)}}$. Thus the principally quasi-polarized 2-divisible group of $(\mathcal{A}, \Lambda)_{\mathcal{N}_{W(k)}}$ is of the form

$$(\mathcal{E}^s, \Lambda_{\mathcal{E}^s}).$$

Let $b_{\mathcal{E}} : \mathcal{E} \tilde{\to} \mathcal{E}^{\mathrm{t}}$ be the isomorphism that corresponds naturally to $b_{\mathcal{V}}$. As $b_{\mathcal{V}}$ modulo $2\mathbf{Z}_2$ is alternating, the generic fibre of $b_{\mathcal{E}}[2] : \mathcal{E}[2] \tilde{\to} \mathcal{E}[2]^{\mathrm{t}}$ is a principal quasi-polarization. Thus $b_{\mathcal{E}}[2]$ itself is a principal quasi-polarization.

**3.5.4. Lemma.** *No simple factor of $(\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} \mathbf{F}, \mathcal{I})$ is of orthogonal first type.*

*Proof:* Let $(\mathrm{End}(W(\mathbf{F})^{2n}), \mathcal{I})$ be a simple factor of $(\mathcal{B}_1 \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F}), \mathcal{I})$; it is of orthogonal first type (see Subsubsection 3.5.1). Let $\mathfrak{B}'_n$ be a perfect, symmetric bilinear form on $W(\mathbf{F})^{2n}$ that defines the involution $\mathcal{I}$ of $\mathrm{End}(W(\mathbf{F})^{2n})$, cf. Lemma 3.3.1 (a) and (c). Let $V$ and $K$ be as in Subsection 2.1 and such that there exists a $K$-linear isomorphism $(W(\mathbf{F})^{2n} \otimes_{W(\mathbf{F})} K, \mathfrak{B}'_n) \tilde{\to} (\mathcal{L}_n \otimes_{\mathbf{Z}_{(2)}} K, \mathfrak{B}_n)$, to be viewed as an identification. As $G_{\mathbf{Z}_{(2)}}$ is a reductive group scheme over $\mathbf{Z}_{(2)}$ (cf. property 1.3 (iv)), the Zariski closure of $\mathcal{D}_{nK}$ in $\mathbf{GL}_{W(\mathbf{F})^{2n} \otimes_{W(\mathbf{F})} V}$ is a group scheme isomorphic to a direct factor of $G^0_V = G^{\mathrm{der}}_V$ (see Subsubsection 3.5.1 for the case $V = W(\mathbf{F})$) and thus it is reductive. From this and Lemma 3.2 we get that the reduction modulo $2V$ of the perfect, symmetric bilinear form $\mathfrak{B}'_n$ on $W(\mathbf{F})^{2n} \otimes_{W(\mathbf{F})} V$ is alternating. Thus $\mathfrak{B}'_n$ modulo $2W(\mathbf{F})$ is alternating. $\square$

**3.5.5. Lemma.** *Let $l$ be an algebraically closed field whose characteristic is either* 0 *or* 2*. Let $\mathcal{L}_n \otimes_{\mathbf{Z}_{(2)}} l = F^1_l \oplus F^0_l$ be a direct sum decomposition such that we have $\mathfrak{Q}_n(x) = 0$ for all $x \in F^1_l \cup F^0_l$. Then the normalizer $\mathcal{P}_{nl}$ of $F^1_l$ in $\mathcal{G}_{nl}$ is a parabolic subgroup of $\mathcal{G}_{nl}$ and we have* $\dim(\mathcal{G}_{nl}/\mathcal{P}_{nl}) = \frac{n(n-1)}{2}$.

*Proof:* We can assume that $n \geq 2$. Let $\mu_l : \mathbf{G}_{ml} \to \mathcal{G}_{nl}$ be the cocharacter that fixes $F^0_l$ and that acts as the inverse of the identity character of $\mathbf{G}_{ml}$ on $F^1_l$. Let $T_l$ be a

maximal torus of $\mathcal{G}_{nl}$ through which $\mu_l$ factors; it is also a torus of $\mathcal{P}_{nl}$. Let $\mathrm{Lie}(\mathcal{G}_{nl}) = \mathrm{Lie}(T_l)\bigoplus_{\alpha\in\Theta}\mathfrak{g}_\alpha$ be the root decomposition relative to $T_l$; thus $\Theta$ is a root system of $D_n$ Lie type. Let $\Theta_0$ be the subset of $\Theta$ formed by roots $\alpha$ with the property that $\mu_l$ acts on $\mathfrak{g}_\alpha$ either trivially or via the inverse of the identity character of $\mathbf{G}_{ml}$. As $\mu_l$ factors through $T_l$, we have $\Theta = \Theta_0 \cup (-\Theta_0)$. If we have $\alpha_1, \alpha_2, \alpha_1+\alpha_2 \in \Theta$, then $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{\alpha_2}] = \mathfrak{g}_{\alpha_1+\alpha_2}$ (cf. Chevalley rule of [12, Vol. III, Exp. XXIII, Cor. 6.5] applied to the $D_n$ Lie type). This implies that $(\Theta_0+\Theta_0)\cap\Theta \subseteq \Theta_0$. Thus $\Theta_0$ is a parabolic subset of $\Theta$. Therefore $\mathrm{Lie}(T_l)\bigoplus_{\alpha\in\Theta_0}\mathfrak{g}_\alpha$ is the Lie algebra of a unique parabolic subgroup $\mathcal{P}_{nl}'$ of $\mathcal{G}_{nl}$ that contains $T_l$, cf. [12, Vol. III, Exp. XXVI, Prop. 1.4]. We have $\mathrm{Lie}(\mathcal{P}_{nl}') = \mathrm{Lie}(\mathcal{P}_{nl})$. The group $\mathcal{P}_{nl}'$ is generated by $T_l$ and by $\mathbf{G}_{al}$ subgroups of $\mathcal{G}_{nl}$ that are normalized by $T_l$ and whose Lie algebras are contained in $\mathrm{Lie}(\mathcal{P}_{nl}') = \mathrm{Lie}(\mathcal{P}_{nl})$. We easily get that each such $\mathbf{G}_{al}$ subgroup is contained in $\mathcal{P}_{nl}$. Thus $\mathcal{P}_{nl}' \leqslant \mathcal{P}_{nl}$. As $\mathrm{Lie}(\mathcal{P}_{nl}') = \mathrm{Lie}(\mathcal{P}_{nl})$, we get that $\dim(\mathcal{P}_{nl}) = \dim_l(\mathrm{Lie}(\mathcal{P}_{nl}))$. Thus $\mathcal{P}_{nl}$ is smooth. Thus as $\mathcal{P}_{nl}' \leqslant \mathcal{P}_{nl}$, the group $\mathcal{P}_{nl}$ is a parabolic subgroup of $\mathcal{G}_{nl}$.

To check that $\dim(\mathcal{G}_{nl}/\mathcal{P}_{nl}) = \frac{n(n-1)}{2}$, it suffices to show that $\Theta \setminus \Theta_0$ has $\frac{n(n-1)}{2}$ roots. If $l$ has characteristic 2, then $\mu_l$ lifts to a cocharacter of $\mathcal{G}_{nW(l)}$ (cf. [12, Vol. II, Exp. IX, Thms. 3.6 and 7.1]) and thus to check that $\Theta \setminus \Theta_0$ has $\frac{n(n-1)}{2}$ roots we can replace $l$ by an algebraic closure of $B(l)$. Therefore we can assume that $l$ has characteristic 0. Even more, we can assume that $l = \mathbf{C}$. If $l = \mathbf{C}$, then the cocharacter $\mu_l : \mathbf{G}_{ml} \to \mathcal{G}_{nl}$ is isomorphic to the cocharacter $\mu_h : \mathbf{G}_{ml} \to G_{\mathbf{C}}$ of Subsubsection 3.5.1. Thus $\mathcal{G}_{nl}/\mathcal{P}_{nl}$ is isomorphic to the compact dual of the hermitian symmetric domain $\mathbf{SO}_{2n}^*(\mathbf{R})/\mathbf{U}_n(\mathbf{R})$ and therefore has dimension $\frac{n(n-1)}{2}$, cf. Subsubsection 3.5.1. Thus $\Theta \setminus \Theta_0$ has $\frac{n(n-1)}{2}$ roots if $l = \mathbf{C}$. $\square$

## 4. Crystalline notations and basic properties

Until the end we will use the notations of Subsections 1.3, 2.1, 3.5, and 3.5.2 and we will take the algebraically closed field $k$ to have a countable transcendental degree over $\mathbf{F}_2$. For $b \in \mathcal{B} \subseteq \mathcal{J}$, we denote also by $b$ the $\mathbf{Z}_{(2)}$-endomorphism of $\mathcal{A}_{E(G,\mathcal{X})}$ defined naturally by $b$ (cf. the classical moduli interpretation of $\mathcal{N}_{E(G,\mathcal{X})}$; see [25, Section 5]). As $\mathcal{N}$ is a closed subscheme of $\mathcal{M}$, it is easy to see that each invertible element of $\mathcal{B}$ extends uniquely to a $\mathbf{Z}_{(2)}$-automorphism of $\mathcal{A}$. As each element $b \in \mathcal{B}$ is a sum of two invertible elements of $\mathcal{B}$, we get that $b$ extends uniquelly to a $\mathbf{Z}_{(2)}$-endomorphism of $\mathcal{A}$. Thus in what follows we will denote also by $\mathcal{B}$ the $\mathbf{Z}_{(2)}$-algebra of $\mathbf{Z}_{(2)}$-endomorphisms with which $\mathcal{A}$ is naturally endowed and therefore with which each abelian scheme obtained from $\mathcal{A}$ by pull back is endowed. For $b \in \mathcal{B}$, we denote also by $b$ different de Rham (crystalline) realizations of $\mathbf{Z}_{(2)}$-endomorphisms that correspond to $b$. In particular, we will speak about the $\mathbf{Z}_{(2)}$-monomorphism $\mathcal{B}^{\mathrm{opp}} \hookrightarrow \mathrm{End}(M_y)$ that makes $M_y$ to be a $\mathcal{B}^{\mathrm{opp}} \otimes_{\mathbf{Z}_{(2)}} W(k)$-module and makes $M_y^*$ to be a $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$-module (here $\mathcal{B}^{\mathrm{opp}}$ is the opposite $\mathbf{Z}_{(2)}$-algebra of $\mathcal{B}$). In this Section we mainly introduce the basic crystalline setting and properties needed in the proof of the Main Theorem. Let $d := \dim_{\mathbf{C}}(\mathcal{X})$.

**4.1. The basic setting.** The actions of $G(\mathbf{A}_f^{(2)})$ on $\mathrm{Sh}(G,\mathcal{X})/H_2$ and $\mathcal{M}$ give birth

to actions of $G(\mathbf{A}_f^{(2)})$ on both $\mathcal{N}$ and $\mathcal{N}^{\mathrm{n}}$. Let $H_0$ be a compact, open subgroup of $G(\mathbf{A}_f^{(2)})$ such that we have $H_0 \leqslant \mathrm{Ker}(\mathbf{GSp}(L,\psi)(\widehat{\mathbf{Z}}) \to \mathbf{GSp}(L,\psi)(\mathbf{Z}/l\mathbf{Z}))$ for some $l \in 1+2\mathbf{N}$. The group $H_0$ acts freely on $\mathcal{M}$ (see Serre's result [32, Section 21, Thm. 5]) and the quotient scheme $\mathcal{M}/H_0$ exists and is a finite scheme over the Mumford scheme $\mathcal{A}_{\frac{\dim_{\mathbf{Q}}(W)}{2},1,l}$. Thus $H_0$ acts freely on the closed subscheme $\mathcal{N}$ of $\mathcal{M}_{O_{(v)}}$ and therefore also on $\mathcal{N}^{\mathrm{n}}$; moreover the quotient schemes $\mathcal{N}/H_0$ and $\mathcal{N}^{\mathrm{n}}/H_0$ exist. Thus $\mathcal{N}$ and $\mathcal{N}^{\mathrm{n}}$ are pro-étale covers of $\mathcal{N}/H_0$ and $\mathcal{N}^{\mathrm{n}}/H_0$ (respectively). Both $\mathcal{N}/H_0$ and $\mathcal{N}^{\mathrm{n}}/H_0$ are flat $O_{(v)}$-schemes of finite type whose generic fibres are smooth (see [38, proof of Prop. 3.4.1]). Thus $\mathcal{N}^{\mathrm{s}}$ is the pull back to $\mathcal{N}^{\mathrm{n}}$ of the smooth locus $\mathcal{N}^{\mathrm{s}}/H_0$ of $\mathcal{N}^{\mathrm{n}}/H_0$ and moreover we have $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})} = \mathcal{N}_{E(G,\mathcal{X})}$. But $\mathcal{N}^{\mathrm{s}}/H_0$ is an open subscheme of $\mathcal{N}^{\mathrm{n}}/H_0$ and thus $\mathcal{N}^{\mathrm{s}}$ is an open subscheme of $\mathcal{N}^{\mathrm{n}}$ that is $G(\mathbf{A}_f^{(2)})$-invariant. As $\mathcal{N}/H_0$ is an excellent scheme (see [27, Subsections (34.A) and (34.B)]), the morphism $\mathcal{N}^{\mathrm{n}}/H_0 \to \mathcal{N}/H_0$ of $O_{(v)}$-schemes is finite. Thus the morphism $\mathcal{N}^{\mathrm{n}} \to \mathcal{N}$ of $O_{(v)}$-schemes is also finite. The relative dimensions of $\mathcal{N}/H_0$ and $\mathcal{N}^{\mathrm{n}}/H_0$ over $O_{(v)}$ are equal to $d$.

**4.1.1. The difficulty.** We would like to show that each point $y \in \mathcal{N}_{W(k)}(k)$ factors through $\mathcal{N}^{\mathrm{s}}_{W(k)}$. As we are in the case (D), the difficulty we face is that the formal deformation space $\mathfrak{D}_y$ over $\mathrm{Spf}(k)$ of the triple $(A, \lambda_A, \mathcal{B})$ is not formally smooth over $\mathrm{Spf}(k)$ of dimension $d$. One can check this starting from the fact (see the end of the fourth paragraph of Subsubsection 3.5.1) that the dimension of the subgroup of $\mathbf{GSp}(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{F}_2, \psi)$ that centralizes $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} \mathbf{F}_2$ is greater than $\dim(G)$. We detail this in the case when $G_{\mathbf{Z}_2}$ is isomorphic to $\mathbf{GSO}^+_{2n\mathbf{Z}_2}$. One always expects that Theorem 1.4 (a) holds and thus implicitly that the centralizer of $\mathcal{B}^{\mathrm{opp}} \otimes_{\mathbf{Z}_{(2)}} W(k)$ in $\mathbf{Sp}(M_y, \lambda_{M_y})$ has a special fibre $C_k$ that is an $\mathbf{Sp}_{2nk}$ group. If the special fibre $C_k$ is an $\mathbf{Sp}_{2nk}$ group, then the deformation theories of Subsubsection 1.2.3 imply that the tangent space of $\mathfrak{D}_y$ has dimension $\frac{n(n+1)}{2}$ which is greater than $d = \frac{n(n-1)}{2}$ (cf. end of Subsubsection 3.5.1 for the last equality).

Therefore the deformation theories of Subsubsection 1.2.3 do not suffice to show that $\mathcal{N}^{\mathrm{n}}_{W(k)}$ is formally smooth over $W(k)$ at points above $y \in \mathcal{N}_{W(k)}(k)$. This explains why in what follows we will use heavily Section 3 as well as several other crystalline theories and new ideas.

**4.1.2. Quasi-sections.** The flat morphism $\mathcal{N}_{W(k)}/H_0 \to \mathrm{Spec}(W(k))$ is of finite type. Thus it has quasi-sections, cf. [18, Cor. (17.16.2)]. This implies that there exists a lift

$$z : \mathrm{Spec}(V) \to \mathcal{N}_{W(k)}$$

of $y : \mathrm{Spec}(k) \to \mathcal{N}_{W(k)}$, where $V$ is a discrete valuation ring as in Subsection 2.1. We emphasize that $z$ is not necessarily a closed embedding. We fix an algebraic closure $\bar{K}$ of $K := V[\frac{1}{2}]$. Let

$$(A_V, \lambda_{A_V}) := z^*((\mathcal{A}, \Lambda)_{\mathcal{N}_{W(k)}}).$$

We denote also by $\lambda_{A_V}$ the perfect form on $(H^1_{dR}(A_V/V))^*$ that is the de Rham realization of $\lambda_{A_V}$. Let $F^1_V$ be the Hodge filtration of $H^1_{dR}(A_V/V)$ defined by $A_V$. Under the

canonical identification $H^1_{dR}(A_V/V)\otimes_V k = M_y/2M_y$, the $k$-vector space $F^1_V\otimes_V k$ gets identified with the kernel of the reduction of $\Phi_y$ modulo $2W(k)$.

We fix an $O_{(v)}$-monomorphism $i_V : V \hookrightarrow \mathbf{C}$. Let $w=[h,g_w]\in\mathcal{N}(\mathbf{C})=\mathcal{N}_{W(k)}(\mathbf{C})$ be the composite of $\mathrm{Spec}(\mathbf{C})\to\mathrm{Spec}(V)$ with $z$; thus $A_{\mathbf{C}}$ is $w^*(\mathcal{A})$ of Subsubsection 3.5.2. The standard functorial isomorphism between the Betti and de Rham homologies of $A_{\mathbf{C}}$ and the identification $L_{(2)}=(H^1_{\acute{e}t}(A_{\mathbf{C}},\mathbf{Z}_{(2)}))^*$ of Subsubsection 3.5.2, give birth to an isomorphism

$$i_{A_{\mathbf{C}}} : L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{C}\tilde{\to}(H^1_{dR}(A_V/V))^*\otimes_V\mathbf{C}$$

of $\mathcal{B}$-modules that takes $\psi$ to $\gamma\lambda_{A_V}$ for some $\gamma\in\mathbf{G}_m(\mathbf{C})$.

**4.2. Fontaine theory.** Let $\tilde{K}$ be a finite field extension of $K$ contained in $\bar{K}$. Let $\tilde{V}$ be the ring of integers of $\tilde{K}$. Let $\tilde{B}$ be the Fontaine ring of $\tilde{K}$ as used in [17]; it is a commutative, integral $B(k)$-algebra equipped with a Frobenius endomorphism and a $\mathrm{Gal}(\bar{K}/\tilde{K})$-action. Let $\tilde{z}\in\mathcal{N}_{W(k)}(\tilde{V})$ be the composite of the natural morphism $\mathrm{Spec}(\tilde{V})\to\mathrm{Spec}(V)$ of $W(k)$-schemes with $z$. We identify $L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2=(H^1_{\acute{e}t}(A_V\times_V\bar{K},\mathbf{Z}_2))^*$ with the Tate-module of the 2-divisible group of $A_{\tilde{K}}$. Under this identification, [17, Thm. 6.2] provides us with a functorial $\tilde{B}$-linear isomorphism

$$c_y : L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\tilde{B}\tilde{\to}M_y^*\otimes_{W(k)}\tilde{B}$$

that preserves all structures. We choose $\tilde{K}$ such that the tensor $v_\alpha\in\mathcal{T}(L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Q}_2)$ of Subsubsection 3.5.2 is fixed by the natural action of $\mathrm{Gal}(\bar{K}/\tilde{K})$ on $L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Q}_2 = (H^1_{\acute{e}t}(A_V\times_V\bar{K},\mathbf{Q}_2))^*$. [One can check that we can always take $\tilde{K}$ to be $K$; but this is irrelevant for what follows.] From Fontaine comparison theory we get that $c_y$ takes $v_\alpha\in\mathcal{T}(W)$ to a tensor $t_\alpha\in\mathcal{T}(M_y^*[\frac{1}{2}])$ and takes $\psi$ to a $\mathbf{G}_m(\tilde{B})$-multiple of the form $\lambda_{M_y}$ on $M_y^*\otimes_{W(k)}\tilde{B}$. If $b\in\mathcal{B}\subseteq\mathcal{J}$, then $t_b$ is the crystalline realization $b$ of the $\mathbf{Z}_{(2)}$-endomorphism $b$ of $A$. From the property 1.3 (iv) we get that the group $G_{B(k)}$ is split. From this and the existence of the cocharacter $\mu_h$ of Subsection 3.5, we get that $G_{B(k)}$ has cocharacters that act on the $B(k)$-span of $\psi$ via the identity character of $\mathbf{G}_{m\,B(k)}$. Thus by composing $c_y$ with a $\tilde{B}$-valued point of the image of such a cocharacter, we get the existence of a symplectic isomorphism

$$c_y' : (L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\tilde{B},\psi)\tilde{\to}(M_y^*\otimes_{W(k)}\tilde{B},\lambda_{M_y})$$

that takes each $v_\alpha$ to $t_\alpha$. As the group $G^0=G\cap\mathbf{Sp}(W,\psi)$ is connected (see Subsection 3.5), the only class in $H^1(\mathrm{Gal}(\bar{K}/B(k)),G^0_{B(k)})$ is the trivial one (see [35, Ch. II, Subsection 3.3 and Ch. III, Subsection 2.3]). From this and the existence of $c_y'$, we get the existence of a $B(k)$-linear isomorphism

$$j_y : L_{(2)}\otimes_{\mathbf{Z}_{(2)}}B(k)\tilde{\to}M_y^*[\frac{1}{2}] \tag{5}$$

that takes $\psi$ to $\lambda_{M_y}$ and takes $v_\alpha$ to $t_\alpha$ for all $\alpha\in\mathcal{J}$.

**4.3. A crystalline form of 3.5.3.** Until the end we assume that $G_{\mathbf{Z}_2}$ is isomorphic to $\mathbf{GSO}_{2n}{}^{+}_{\mathbf{Z}_2}$. Thus $d = \frac{n(n-1)}{2}$, cf. Subsubsection 3.5.1. Let $s \in \mathbf{N} \setminus \{1\}$ and $(\mathcal{V}, b_{\mathcal{V}})$ be as in Subsubsection 3.5.3. Let

$$(\mathcal{E}^s_z, \lambda_{\mathcal{E}^s_z}) \tag{6}$$

be the pull back of $(\mathcal{E}^s, \Lambda_{\mathcal{E}^s})$ (see Subsubsection 3.5.3) through $z$; it is the principally quasi-polarized 2-divisible group of $(A_V, \lambda_{A_V})$. Let $b_z : \mathcal{E}_z \tilde{\to} \mathcal{E}^{\mathrm{t}}_z$ be the pull back through $z$ of the isomorphism $b_{\mathcal{E}}$ of Subsubsection 3.5.3. To (6) corresponds a direct sum decomposition

$$(M_y, \Phi_y) = (N^s_y, \Phi_y). \tag{7}$$

Let $b_{N_y}$ be the perfect, symmetric bilinear form on $N_y$ that correspond to $b_z$ (via the Dieudonné functor $\mathbf{D}$). Let $(\bar{N}_y, b_{\bar{N}_y}) := (N_y, b_{N_y}) \otimes_{W(k)} k$. From the end of Subsubsection 3.5.3 we get that:

**4.3.1. Fact.** *The isomorphism $b_z[2] : \mathcal{E}_z[2] \tilde{\to} \mathcal{E}_z[2]^{\mathrm{t}}$ is a principal quasi-polarization of $\mathcal{E}_z[2]$.*

## 5. Proof of 1.4 (a)

We will use the notations of Subsections 1.3, 2.1, 3.5, 3.5.2, and 3.5.3. In this Section we prove Theorem 1.4 (a). The hard part of the proof is to show that Lemma 3.5.4 transfers naturally to the corresponding crystalline contexts that pertain to $y$; the below Theorem surpasses (in the geometric context of the Main Theorem) the problem (i) of Subsection 1.1.

**5.1. Theorem.** *Suppose that $G_{\mathbf{Z}_2}$ is isomorphic to $\mathbf{GSO}_{2n}{}^{+}_{\mathbf{Z}_2}$. Then for each point $y \in \mathcal{N}_{W(k)}(k)$, the perfect bilinear form $b_{\bar{N}_y}$ on $\bar{N}_y$ is alternating (see Subsection 4.3 for notations).*

*Proof:* We recall $b_{N_y}$ is symmetric. If $V = W(k)$, then based on Fact 4.3.1 the Theorem follows from Lemma 2.2.3 applied to $(\mathcal{E}_z[2], b_z[2])$. The proof of the Theorem in the general case (of an arbitrary index $[V : W(k)] \in \mathbf{N}$) is a lot more harder and therefore below we will number its main parts. Let $\mathrm{Spec}(S)$ be a connected étale cover of an affine, connected, open subscheme of the reduced scheme of $\mathcal{N}_k/H_0$, which is smooth over $k$ and which specializes to the $k$-valued point of $\mathcal{N}_k/H_0$ defined by $y$. In Subsubsections 5.1.1 to 5.1.5 we will use only properties 1.3 (i) and (iii); but in Subsubsection 5.1.6 we will use also the property 1.3 (iv) and thus implicitly Lemma 3.5.4.

**5.1.1. Part I: notations.** We can assume that the compact, open subgroup $H_0$ of $G(\mathbf{A}^{(2)}_f)$ is such that: (i) the triple $(\mathcal{A}, \Lambda, \mathcal{B})$ is the pull back of a similar triple $(\mathcal{A}_{H_0}, \Lambda_{H_0}, \mathcal{B})$ over $\mathcal{N}/H_0$ (this holds if $H_0 \leqslant \mathrm{Ker}(\mathbf{GSp}(L, \psi)(\widehat{\mathbf{Z}}) \to \mathbf{GSp}(L, \psi)(\mathbf{Z}/l\mathbf{Z}))$ for some odd $l >> 0$), (ii) we have a direct sum decomposition $\mathcal{A}_{H_0}[2] \times_{\mathcal{N}/H_0} S = E^s$ that

corresponds naturally to the decomposition (see Subsubsection 3.5.3) $\mathcal{A}_{\mathcal{N}_{W(k)}}[2] = (\mathcal{E}[2])^s$, and (iii) there exists an isomorphism $\lambda_E : E \tilde{\to} E^{\mathrm{t}}$ that corresponds naturally to $b_{\mathcal{V}}$. From Fact 4.3.1 we get that the isomorphism $\lambda_E$ is a principal quasi-polarization. Let

$$(N, \phi, \upsilon, \nabla, b_N)$$

be the evaluation of $\mathbf{D}(E, \lambda_E)$ at the trivial thickening of $\mathrm{Spec}(S)$; thus $N$ is a projective $S$-module. Let $F := \mathrm{Ker}(\phi)$; it is the Hodge filtration of $N$ defined by $E$. We consider the $S$-submodule

$$X := \{x \in N | b_N(x, x) = 0\} \subseteq N.$$

To prove the Theorem, using specializations we get that it suffices to show that $b_N$ is alternating. To check this thing, we will often either shrink $S$ (i.e., we will replace $\mathrm{Spec}(S)$ by an open, dense subscheme of it) or replace $\mathrm{Spec}(S)$ by a connected étale cover of it. We will show that the assumption that $b_N$ is not alternating, leads to a contradiction (the argument will end up before Subsection 5.2).

As $b_N(F, F) = 0$, we have $F \subseteq X$. By shrinking $S$, we can assume that $N$ and $F$ are free $S$-modules, that the reduction of $b_N$ modulo each maximal ideal of $S$ is not alternating, and that we have a short exact sequence of nontrivial free $S$-modules

$$0 \to X \to N \to N/X \to 0. \tag{8}$$

We have $\mathrm{rk}_S(N) = 2n$ and (cf. the existence of $\lambda_E$) $\mathrm{rk}_S(F) = n$. Let $r := \mathrm{rk}_S(N/X)$. We have $r \in \{n, \ldots, 2n-1\}$ and $\mathrm{rk}_S(X) = 2n - r$. As the field of fractions of $S$ is not perfect, we can not use Fact 3.1.1 (a) and (b) to get directly that $r$ is 1.

**5.1.2. Part II: Dieudonné theory.** The short exact sequence (8) is preserved by $\phi$ and $\upsilon$. Even more, we have $\phi(N^{(2)}) \subseteq X$ and $\upsilon(N) \subseteq X^{(2)}$. Thus the resulting Frobenius and Verschiebung maps of $N/X$ are both 0 maps. By shrinking $S$, we can assume that $S$ has a finite 2-basis. From the fully faithfulness part of [5, Thm. 4.1.1 or 4.2.1] we get that to (8) corresponds a short exact sequence

$$0 \to \boldsymbol{\alpha}_2{}_S^r \to E \to E_1 \to 0$$

of commutative, finite, flat group schemes over $S$. Let

$$E_2 := \lambda_E^{-1}(E_1^{\mathrm{t}}) \leqslant \lambda_E^{-1}(E^{\mathrm{t}}) = E.$$

The quotient group scheme $E/E_2$ is isomorphic to $(\boldsymbol{\alpha}_2{}_S^r)^{\mathrm{t}}$ and therefore to $\boldsymbol{\alpha}_2{}_S^r$. Let $Y$ be the direct summand of $N$ that defines the evaluation of $\mathrm{Im}(\mathbf{D}(E/E_2) \to \mathbf{D}(E))$ at the trivial thickening of $S$. As $E/E_2 \tilde{\to} \boldsymbol{\alpha}_2{}_S^r$, we have $\phi(Y \otimes 1) = 0$. Thus $Y \subseteq F$ and therefore the $S$-linear map $Y \to N/X$ is 0. This implies that the homomorphism $\boldsymbol{\alpha}_2{}_S^r \to E/E_2$ over $S$ is trivial, cf. [5, Thm. 4.1.1 or 4.2.1]. Thus the subgroup scheme $\boldsymbol{\alpha}_2{}_S^r$ of $E$ is in fact a subgroup scheme of $E_2$.

The perpendicular of $Y$ with respect to $b_N$ is $X$, cf. constructions. Thus $b_N$ induces naturally a perfect form $b_{\tilde{N}}$ on $\tilde{N} := X/Y$ that (cf. the definition of $X$) is alternating. Let

$$\tilde{E} := E_2/\boldsymbol{\alpha}_{2}{}_S^r.$$

Let $\lambda_{\tilde{E}} : \tilde{E} \tilde{\to} \tilde{E}^{\mathrm{t}}$ be the isomorphism of finite, flat group schemes over $S$ induced by $\lambda_E$. The evaluation of $\mathbf{D}(\tilde{E}, \lambda_{\tilde{E}})$ at the trivial thickening of $\mathrm{Spec}(S)$ is $(\tilde{N}, \tilde{\phi}, \tilde{v}, \tilde{\nabla}, b_{\tilde{N}})$, where $(\tilde{\phi}, \tilde{v}, \tilde{\nabla})$ is obtained from $(\phi, v, \nabla)$ via a natural passage to quotients.

**5.1.3. Part III: computing $r$.** As $S$ is an étale cover of a locally closed subscheme of $\mathcal{M}_k/H_0$, the image of the Kodaira–Spencer map $\mathcal{H}$ of $\nabla$ (i.e., of $E$) is a direct summand of $\mathrm{Hom}(F, N/F)$; by shrinking $S$ we can assume that this direct summand is free of rank $d = \frac{n(n-1)}{2}$. The connection $\nabla$ restricts to connections on $Y$ as well as on $X$. From this and the inclusions $Y \subseteq F \subseteq X$, we get that $\mathrm{Im}(\mathcal{H})$ is a direct summand of the direct summand $\mathrm{Hom}(F/Y, X/F)$ of $\mathrm{Hom}(F, N/F)$ and thus it is canonically identified with the image of the Kodaira–Spencer map $\tilde{\mathcal{H}}$ of $\tilde{E}$. As $b_{\tilde{N}}$ is alternating, we have $\mathrm{rk}_S(\mathrm{Im}(\tilde{\mathcal{H}})) \leq \frac{(n-r)(n-r+1)}{2} \leq d$. [We recall that the moduli scheme $\mathcal{A}_{n-r,1,l}$ (see Subsubsection 1.2.1) has relative dimension $\frac{(n-r)(n-r+1)}{2}$.] Thus the three numbers $d = \frac{n(n-1)}{2}$, $\frac{(n-r)(n-r+1)}{2}$, and $\mathrm{rk}_S(\mathrm{Im}(\tilde{\mathcal{H}})) = \mathrm{rk}_S(\mathrm{Im}(\mathcal{H}))$ must be equal. Thus $r = 1$ and the pair $(\tilde{E}, \lambda_{\tilde{E}})$ is a versal deformation at each $k$-valued point of $\mathrm{Spec}(S)$. As $r = 1$ and $n \geq 2$, we have $\tilde{N} \neq 0$.

**5.1.4. Part IV: reduction to an ordinary context.** We check that by shrinking $S$ and by passing to an étale cover of $S$, we can assume that there exists a short exact sequence

$$0 \to \boldsymbol{\mu}_{2}{}_S^{n-1} \to \tilde{E} \to \mathbf{Z}/2\mathbf{Z}_S^{n-1} \to 0. \tag{9}$$

This well known property is a consequence of the versality part of the end of Subsubsection 5.1.3. For the sake of completeness, we will include a self contained argument for (9).

Let $\tilde{F} := F/Y \subseteq \tilde{N}$. Let $J$ be a fixed maximal ideal of $S$. Let $(\tilde{N}_k, \tilde{\phi}_k, \tilde{v}_k, \tilde{F}_k, b_{\tilde{N}_k})$ be the reduction of $(\tilde{N}, \tilde{\phi}, \tilde{v}, \tilde{F}, b_{\tilde{N}})$ modulo $J$. By shrinking $S$, we can assume that $\tilde{N}$ and $\tilde{F}$ are free $S$-modules. We fix an isomorphism

$$I_S : (\tilde{N}_k, \tilde{F}_k, b_{\tilde{N}_k}) \otimes_k S \tilde{\to} (\tilde{N}, \tilde{F}, b_{\tilde{N}})$$

of filtered symplectic spaces over $S$. Let $\tilde{N}_k = \tilde{F}_k \oplus \tilde{Q}_k$ be a direct sum decomposition such that $\tilde{Q}_k$ is a maximal isotropic subspace of $\tilde{N}_k$ with respect to $b_{\tilde{N}_k}$. We define $\tilde{H} := \mathbf{Sp}(\tilde{N}_k, b_{\tilde{N}_k})$. Any two standard symplectic $S$-bases of $\tilde{N}_k \otimes_k S$ with respect to $b_{\tilde{N}_k}$ are $\tilde{H}(S)$-conjugate. Thus there exist elements $h_S$, $\tilde{h}_S \in \tilde{H}(S)$ such under $I_S$, $\tilde{\phi}$ and $\tilde{v}$ become isomorphic to $h_S \circ (\tilde{\phi}_k \otimes 1_S)$ and $(\tilde{v}_k \otimes 1_S) \circ \tilde{h}_S^{-1} h_S^{-1}$ (respectively). As $\tilde{v} \circ \tilde{\phi} = 0$, we get that $\tilde{h}_S$ normalizes $\mathrm{Im}(\tilde{\phi}_k) \otimes_k S = \mathrm{Ker}(\tilde{v}_k) \otimes_k S$. As for $x \in \tilde{N}_k^{(2)}$ and $u \in \tilde{N}_k$ we have identities $b_{\tilde{N}_k}(h_S(\tilde{\phi}_k(x \otimes 1)), u \otimes 1) = b_{\tilde{N}_k}(x \otimes 1, (\tilde{v}_k \otimes 1_S)\tilde{h}_S^{-1} h_S^{-1}(u \otimes 1))$ and

$b_{\tilde{N}_k}(\tilde{\phi}_k(x\otimes 1)),u) = b_{\tilde{N}_k}(x\otimes 1,\tilde{v}_k(u))$, we get $b_{\tilde{N}_k}(\tilde{\phi}_k(x\otimes 1),\tilde{h}_S(u\otimes 1)-u\otimes 1)=0$; thus $\tilde{h}_S(u\otimes 1)-u\otimes 1\in \mathrm{Im}(\tilde{\phi}_k)\otimes_k S = \mathrm{Ker}(\tilde{v}_k)\otimes_k S$. These two properties of $\tilde{h}_S$ imply that $(\tilde{v}_k\otimes 1_S)\circ \tilde{h}_S^{-1} = \tilde{v}_k\otimes 1_S$. Thus we can assume that $\tilde{h}_S$ is the identity element of $\tilde{H}(S)$.

Let $\tilde{h}_o\in \tilde{H}(k)$ be such that $\tilde{h}_o\tilde{\phi}_k((\tilde{Q}_k)^{(2)}) = \tilde{Q}_k$; this implies that $(\tilde{N}_k,\tilde{h}_o\tilde{\phi}_k,\tilde{v}_k\tilde{h}_o^{-1})$ is an ordinary Dieudonné module over $k$. Thus there exists a non-empty open subscheme $\tilde{\mathcal{O}}$ of $\tilde{H}$ such that for each element $\tilde{g}\in\tilde{\mathcal{O}}(k)$, the triple $(\tilde{N}_k,\tilde{g}\circ\tilde{\phi}_k,\tilde{v}_k\circ\tilde{g}^{-1})$ is an ordinary Dieudonné module over $k$. Let $\tilde{P}$ be the parabolic subgroup of $\tilde{H}$ that normalizes $\tilde{F}_k$. Let $\tilde{C}$ be the Levi subgroup of $\tilde{P}$ that normalizes $\tilde{Q}_k$. For $\tilde{h}\in\tilde{P}(k)$, we write $\tilde{h}=\tilde{h}^u\tilde{c}$, where $\tilde{h}^u$ is a $k$-valued point of the unipotent radical of $\tilde{P}$ and where $\tilde{c}\in\tilde{C}(k)$. We consider the morphism

$$j_S:\tilde{P}\times_k \mathrm{Spec}(S)\to\tilde{H}$$

that takes $(\tilde{h},\tilde{m})\in\tilde{P}(k)\times\mathrm{Spec}(S)(k)$ to $\tilde{u}:=\tilde{h}(h_S\circ\tilde{m})((\tilde{c})^{-1}\circ\sigma)\in\tilde{H}(k)$. As $\mathrm{Im}(\tilde{\mathcal{H}})$ has rank $d=\frac{n(n-1)}{2}$, the composite of $h_S:\mathrm{Spec}(S)\to\tilde{H}$ with the quotient epimorphism $\tilde{H}\twoheadrightarrow\tilde{H}/\tilde{P}$ is smooth. Thus $j_S$ induces $k$-linear isomorphisms at the level of tangent spaces of $k$-valued points. Therefore $j_S^*(\tilde{\mathcal{O}})$ is a non-empty open subscheme of $\tilde{P}\times_k\mathrm{Spec}(S)$. Thus there exists a pair $(\tilde{h},\tilde{m})\in\tilde{P}(k)\times\mathrm{Spec}(S)(k)$ such that we have $\tilde{u}\in\tilde{\mathcal{O}}(k)$. The reduction of $(\tilde{N},\tilde{\phi},\tilde{v})$ modulo the maximal ideal $\tilde{J}$ of $S$ that defines $\tilde{m}$, is isomorphic to $(\tilde{N}_k,(h_S\circ\tilde{m})\tilde{\phi}_k,\tilde{v}_k(h_S\circ\tilde{m})^{-1})$ and therefore also to $(\tilde{N}_k,\tilde{h}(h_S\circ\tilde{m})\tilde{\phi}_k\tilde{h}^{-1},\tilde{h}\tilde{v}_k(h_S\circ\tilde{m})^{-1}\tilde{h}^{-1}) = (\tilde{N}_k,\tilde{u}\tilde{\phi}_k,\tilde{v}_k\tilde{u}^{-1})$. Thus the reduction of $\tilde{E}$ modulo $\tilde{J}$ is an ordinary truncated Barsotti–Tate group of level 1 over $k$. Therefore, by shrinking $S$, we can assume that $\tilde{E}$ is an ordinary truncated Barsotti–Tate group of level 1 over $k$. Thus the short exact sequence (9) exists.

**5.1.5. Part V: filtrations.** Due to (9) and the existence of the short exact sequence $0\to\tilde{E}\to E_1\to\boldsymbol{\alpha}_{2S}\to 0$, we have naturally another short exact sequence $0\to{\boldsymbol{\mu}_{2S}}^{n-1}\to E_1\to\boldsymbol{\alpha}_{2S}\times_S\mathbf{Z}/2\mathbf{Z}_S^{n-1}\to 0$. Due to (9) and the existence of a short exact sequence $0\to\boldsymbol{\alpha}_{2S}\to E_2\to\tilde{E}\to 0$, we get that we have naturally another short exact sequence $0\to\boldsymbol{\alpha}_{2S}\times_S{\boldsymbol{\mu}_{2S}}^{n-1}\to E_2\to\mathbf{Z}/2\mathbf{Z}_S^{n-1}\to 0$. Thus $E$ has a filtration by finite, flat subgroup schemes

$$0\leqslant E_{(1)}\leqslant E_{(2)}\leqslant E, \tag{10}$$

where $E_{(1)}$ is ${\boldsymbol{\mu}_{2S}}^{n-1}$, $E_{(2)}/E_{(1)}$ is the extension of $\boldsymbol{\alpha}_{2S}$ by $\boldsymbol{\alpha}_{2S}$, and $E/E_{(2)}$ is $\mathbf{Z}/2\mathbf{Z}_S^{n-1}$.

In order to benefit from previous notations, until Subsection 5.2 we choose $y:\mathrm{Spec}(k)\to\mathcal{N}_{W(k)}$ such that it defines a morphism $\mathrm{Spec}(k)\to\mathcal{N}_{W(k)}/H_0$ that factors through $\mathrm{Spec}(S)$. Let $\mathcal{E}_y:=\mathcal{E}_z\times_V k$. The 2-ranks of $\mathcal{E}_y$ and $\mathcal{E}_y^{\mathrm{t}}$ are $n-1$ (see (10)). It is well known that this implies that $\mathcal{E}_z$ has a filtration by 2-divisible groups

$$0\leqslant\mathcal{E}_{(1)z}\leqslant\mathcal{E}_{(2)z}\leqslant\mathcal{E}_z, \tag{11}$$

where $\mathcal{E}_{(1)z}={\boldsymbol{\mu}_{2^\infty}}_V^{n-1}$, $\mathcal{E}_z/\mathcal{E}_{(2)z}=(\mathbf{Q}_2/\mathbf{Z}_2)_V^{n-1}$, and the 2-divisible group $\mathcal{E}'_z:=\mathcal{E}_{(2)z}/\mathcal{E}_{(1)z}$ over $V$ is connected and has a connected dual. The height of $\mathcal{E}'_z$ is $2=2n-2(n-1)$.

**5.1.6. Part VI: the new pair $(\mathcal{E}'_z, b'_z)$.** The filtration (11) is compatible in the natural way with $b_z$. Let $b'_z : \mathcal{E}'_z \tilde{\to} \mathcal{E}'^t_z$ be the isomorphism induced naturally by $b_z$. Let $T_2(\mathcal{E}'_{zK})$ be the Tate-module of the generic fibre $\mathcal{E}'_{zK}$ of $\mathcal{E}'_z$. As $\mathcal{E}'_z$ has height 2, $T_2(\mathcal{E}'_{zK})$ is a a free $\mathbf{Z}_2$-module of rank 2 on which $\mathrm{Gal}(\bar{K}/K)$-acts. To $b'_z$ corresponds a perfect, symmetric bilinear form (denoted also by) $b'_z$ on $T_2(\mathcal{E}'_{zK})$ that modulo $2\mathbf{Z}_2$ is alternating. Let $\mathbf{GSO'}$ be the Zariski closure in $\mathbf{GL}_{T_2(\mathcal{E}'_{zK})}$ of the identity component of the subgroup of $\mathbf{GL}_{T_2(\mathcal{E}'_{zK})[\frac{1}{2}]}$ that normalizes the $\mathbf{Z}_2$-span of $b'_z$.

As $\mathbf{GSO}'_{W(\mathbf{F})}$ is isomorphic to $\mathbf{GSO}_{2W(\mathbf{F})}$ (cf. Proposition 3.4 (b)), $\mathbf{GSO'}$ is a rank 2 torus. By replacing $(V, z)$ with a pair $(\tilde{V}, \tilde{z})$ as in Subsection 4.2, we can assume that the Galois representation $\mathrm{Gal}(\bar{K}/K) \to \mathbf{GL}_{T_2(\mathcal{E}'_{zK})}(\mathbf{Z}_2)$ factors through $\mathbf{GSO'}(\mathbf{Z}_2)$. Let $\mathcal{B}'$ be the semisimple, commutative $\mathbf{Z}_2$-algebra of endomorphism of $\mathcal{E}'_z$ whose Lie algebra is $\mathrm{Lie}(\mathbf{GSO'})$.

Let $e$ and $R_e$ be as in Subsection 2.1. Let $(N'_z, \Phi'_z, \Upsilon'_z, \nabla'_z, b_{N'_z})$ be the projective limit indexed by $m \in \mathbf{N}$ of the evaluations of $\mathbf{D}((\mathcal{E}'_z, b'_{\mathcal{E}_z})_{U_e})$ at the thickenings attached naturally to the closed embeddings $\mathrm{Spec}(U_e) \hookrightarrow \mathrm{Spec}(R_e/2^m R_e)$. Thus $N'_z$ is a free $R_e$-module of rank 2, we have $R_e$-linear maps $\Phi'_z : N'_z \otimes_{R_e} {}_{\Phi_{R_e}} R_e \to N'_z$ and $\Upsilon'_z : N'_z \to N'_z \otimes_{R_e} {}_{\Phi_{R_e}} R_e$, $\nabla'_z : N'_z \to N'_z \otimes_{R_e} R_e\, dt$ is an integrable and topologically nilpotent connection, and $b_{N'_z}$ is a perfect, symmetric bilinear form on $N'_z$. The $\mathbf{Z}_2$-algebra $\mathcal{B}'$ acts on $N'_z$ and the $R_e \otimes_{\mathbf{Z}_2} \mathcal{B}'$-module $N'_z$ is free of rank 1. To the natural decomposition $\mathcal{B}' \otimes_{\mathbf{Z}_2} W(\mathbf{F}) \tilde{\to} W(\mathbf{F}) \oplus W(\mathbf{F})$ of $W(\mathbf{F})$-algebras corresponds a direct sum decomposition $N'_z = N'^{(1)}_z \oplus N'^{(2)}_z$ into free $R_e$-modules of rank 1. Both $N'^{(1)}_z$ and $N'^{(2)}_z$ are isotropic with respect to $b_{N'_z}$. therefore $b_{N'_z}$ modulo $2R_e$ is alternating, cf. Fact 3.1.1 (a). Thus $(N'_z, b_{N'_z}) \otimes_{R_e} k$ is a symplectic space over $k$ of rank 2.

The fibre of the filtration (11) over $k$ splits. This implies that over $k$ we have a direct sum decomposition

$$(\bar{N}_y, b_{\bar{N}_y}) = (N'_z, b_{N'_z}) \otimes_{R_e} k \oplus (N_o, b_{N_o}), \tag{12}$$

where $(N_o, b_{N_o})$ is part of a quadruple $(N_o, \phi_{N_o}, \upsilon_{N_o}, b_{N_o})$ that is the evaluation at the trivial thickening of $\mathrm{Spec}(k)$ of the Dieudonné functor $\mathbf{D}$ applied to an ordinary truncated Barsotti–Tate group $E_o$ of level 1 over $k$ equipped with an isomorphism $\lambda_{E_o} : E_o \tilde{\to} E_o^{\mathrm{t}}$. It is easy to see that $b_{N_o}$ is an alternating form on $N_o$. Thus $(\bar{N}_y, b_{\bar{N}_y})$ is a symplectic space over $k$, cf. (12). Therefore the reduction of $b_N$ modulo the maximal ideal of $S$ associated naturally to $y$ is alternating. In other words, we reached the desired contradiction. Thus $b_N$ is an alternating form on $N$. This proves the Theorem. $\square$

**5.2. End of the proof of 1.4 (a).** Due to Theorem 5.1, the formula $q_{N_y}(x) := \frac{b_{N_y}(x,x)}{2}$ defines a quadratic form on $N_y$. Let $\tilde{G}_{W(k)}$ be the Zariski closure in $\mathbf{GSp}(M_y, \lambda_{M_y})$ of the identity component of the subgroup of $\mathbf{GSp}(M_y[\frac{1}{2}], \lambda_{M_y})$ that fixes $t_b$ for all $b \in \mathcal{B} \subseteq \mathcal{J}$. We can redefine $\tilde{G}_{W(k)}$ as $\mathbf{GSO}(N_y, q_{N_y})$ and therefore $\tilde{G}_{W(k)}$ is a reductive group scheme, cf. Proposition 3.4 (b).

Let $j_y$ be as in Subsection 4.2. Let $L_y := j_y^{-1}(M_y^*)$; it is a $W(k)$-lattice of $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} B(k)$. We recall that $j_y$ takes $\psi$ to $\lambda_{M_y}$ and takes $b = v_b$ to $b = t_b$ for all $b \in \mathcal{B}$ (cf. (5);

see Subsubsection 3.5.2 and the beginning of Section 4 for these identities). Thus the $W(k)$-lattice $L_y$ of $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} K$ has the following three properties:

**(i)** for all $b \in \mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$ we have $b(L_y) \subseteq L_y$,

**(ii)** the Zariski closure of $G_{B(k)}$ in $\mathbf{GL}_{L_y}$ is a reductive group scheme $j_y^{-1}\tilde{G}_{W(k)}j_y$ over $W(k)$, and

**(iii)** we get a perfect, alternating form $\psi : L_y \otimes_{W(k)} L_y \to W(k)$.

In this paragraph we check that properties (i) to (iii) imply that there exists $g_y \in G^0(B(k))$ such that we have $g_y(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)) = L_y$. Let $T_{W(k)}$ be a maximal split torus of $G_{W(k)}$ which (cf. property (ii)) is also a maximal torus of $j_y^{-1}\tilde{G}_{W(k)}j_y$. The existence of $T_{W(k)}$ follows from the fact that each two points of the building of $G_{B(k)}$, belong to an apartment of the building (see [37, pp. 43–44] for this fact and for the language used). Let $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k) = \oplus_{i\in\chi} W_i$ and $L_y = \oplus_{i\in\chi} W_i'$ be direct sum decompositions indexed by a set $\chi$ of distinct characters of $T_{W(k)}$, such that for all $i \in \chi$ the actions of $T_{W(k)}$ on $W_i$ and $W_i'$ are via the character $i$ of $T_{W(k)}$. The representation of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} B(k)$ on $W_i[\frac{1}{2}] = W_i'[\frac{1}{2}]$ is absolutely irreducible. Thus the representations of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$ on $W_i$ and $W_i'$ are isomorphic and their fibres are absolutely irreducible. Therefore there exists $n_i \in \mathbf{Z}$ such that we have $p^{n_i} W_i = W_i'$. Let $\mu_0 : \mathbf{G}_{mB(k)} \to \mathbf{GL}_{L_{(2)} \otimes_{\mathbf{Z}_{(2)}} B(k)}$ be a cocharacter such that it acts on $W_i[\frac{1}{2}]$ via the $n_i$-th power of the identity character of $\mathbf{G}_{mB(k)}$. Let $g_y := \mu_0(p)$. We have $g_y(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)) = L_y$. As $T_{W(k)}$ is a torus of $\mathbf{GSp}(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k), \psi)$, for each $i \in \chi$ there exists a unique $\tilde{i} \in \chi$ such that for every $i' \in \chi \setminus \{\tilde{i}\}$ we have $\psi(W_i, W_{i'}) = 0$. The map $\chi \to \chi$ that takes $i$ to $\tilde{i}$ is a bijection of order at most 2. But as $L_y$ and $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)$ are both self-dual $W(k)$-lattices of $W \otimes_{\mathbf{Q}} B(k)$ with respect to $\psi$ (cf. property (iii)), for all $i \in \chi$ we have $n_i + n_{\tilde{i}} = 0$. Therefore $\mu_0$ fixes $\psi$. As for $i \in \chi$ the $W(k)$-module $W_i$ is left invariant by $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$, $\mu_0$ fixes all elements $b \in \mathcal{B}$. Thus $\mu_0$ is a cocharacter of $G^0_{1B(k)}$ and therefore also of the identity component $G^0_{B(k)}$ of $G^0_{1B(k)}$. Thus $g_y \in G^0(B(k))$ i.e., $g_y$ exists.

By replacing $j_y$ with $j_y g_y$, we can assume that $j_y(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)) = j_y(L_y) = M_y^*$. Thus $j_y : L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k) \tilde{\to} M_y^*$ is an isomorphism of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$-modules that induces a symplectic isomorphism $(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k), \psi) \tilde{\to} (M_y^*, \lambda_{M_y})$. Thus Theorem 1.4 (a) holds.

## 6. Proof of 1.4 (b)

We recall that until the end we assume that $G_{\mathbf{Z}_2}$ is isomorphic to $\mathbf{GSO}^+_{2n\mathbf{Z}_2}$. Subsections 6.1 to 6.6 are intermediary steps needed to finalize the proof of Theorem 1.4 (b) in Subsection 6.7. In Corollary 6.8 we get a variant of Theorem 1.4 (b) for $\mathcal{N}^{\mathrm{n}}_{k(v)}$. We use the notations of Subsections 4.1 to 4.3. Let $\tilde{G}_{W(k)}$ be the reductive, closed subgroup scheme of $\mathbf{GL}_{M_y}$ introduced in the beginning of Subsection 5.2.

In Subsections 6.1 and 6.2 we show the existence of a good cocharacter $\mu_y : \mathbf{G}_{mW(k)} \to \tilde{G}_{W(k)}$ that produces a direct sum decomposition $M_y = F^1 \oplus F^0$ such that

$F^1$ is the Hodge filtration defined by a lift $(\tilde{A}, \lambda_{\tilde{A}}, \mathcal{B})$ of $(A, \lambda_A, \mathcal{B})$ to $W(k)$. Unfortunately, we can not check directly that we can assume that $(\tilde{A}, \lambda_{\tilde{A}}, \mathcal{B})$ is the pull back of $(\mathcal{A}, \Lambda, \mathcal{B})_{\mathcal{N}_{W(k)}}$ via a $W(k)$-valued point of $\mathcal{N}_{W(k)}$ that lifts $y$ (see Remark 6.2.1; this explains the length of this Section). In Subsection 6.3 we associate to $\mu_y$ a smooth, closed subgroup scheme $\tilde{U}$ of $\tilde{G}_{W(k)}$ of relative dimension $d$ over $W(k)$. In Subsection 6.4 we use $\tilde{U}$ and [14, Section 7] to construct a deformation of $(A, \lambda_A, \mathcal{B})$ that is versal and that defines naturally a morphism $m_{R(d)/2R(d)} : \mathrm{Spec}(R(d)/2R(d)) \to \mathcal{M}_k$ of $k$-schemes, where $R(d)$ is a $W(k)$-algebra of formal power series in $d$ variables. In Subsections 6.5 to 6.7 we use a modulo 2 version of [14, Section 7] to check that $m_{R(d)/2R(d)}$ factors through a formally étale morphism $n_{R(d)/2R(d)} : \mathrm{Spec}(R(d)/2R(d)) \to \mathcal{N}_{k\mathrm{red}}$ of $k$-schemes; the existence of $n_{R(d)/2R(d)}$ will surpass (in the geometric context of the Main Theorem) the modulo $2W(k)$ version of the problems (ii) and (iii) of Subsection 1.1.

**6.1. Proposition.** *There exists a cocharacter $\mu_y : \mathbf{G}_{mW(k)} \to \tilde{G}_{W(k)}$ that gives birth to a direct sum decomposition $M_y = F^1 \oplus F^0$ with the properties that $F^1$ lifts the Hodge filtration $\bar{F}^1_y := F^1_V \otimes_V k$ of $M_y/2M_y$ defined by $A$ and that $\beta \in \mathbf{G}_m(W(k))$ acts on $F^i$ through $\mu_y$ as the multiplication with $\beta^{-i}$ ($i \in \{0,1\}$).*

*Proof:* Let $\tilde{F}^1_0$ be a direct summand of the $W(k)$-module $N_y$ that lifts the Hodge filtration of $\bar{N}_y = N_y/2N_y$. For $u \in \tilde{F}^1_0$ let $x := \frac{\Phi_y(u)}{2} \in N_y$. We have $b_{N_y}(x,x) = \frac{1}{2}\sigma(b_{N_y}(u,u))$ and (cf. Theorem 5.1) $b_{N_y}(x,x) \in 2W(k)$. Thus for all $u \in \tilde{F}^1_0$ we have $b_{N_y}(u,u) \in 4W(k)$. It is easy to see that this implies that we can choose $\tilde{F}^1_0$ such that we have $b_{N_y}(\tilde{F}^1_0, \tilde{F}^1_0) \subseteq 4W(k)$.

Let $\{\tilde{u}_1, \tilde{v}_1, \ldots, \tilde{u}_n, \tilde{v}_n\}$ be a $W(k)$-basis for $N_y$ such that the following three things hold: $\{\tilde{u}_1, \ldots, \tilde{u}_n\}$ is a $W(k)$-basis for $\tilde{F}^1_0$, for each $i \in \{1, \ldots, n\}$ we have $b_{N_y}(\tilde{u}_i, \tilde{v}_i) = 1$, and for each $i,\, j \in \{1, \ldots, n\}$ with $i \neq j$ and for every pair $(u,v) \in \{(\tilde{u}_i, \tilde{v}_j), (\tilde{v}_i, \tilde{v}_j)\}$ we have $b_{N_y}(u,v) \in 4W(k)$. As $b_{\bar{N}_y}$ is alternating (cf. Theorem 5.1) and as $b_{N_y}(\tilde{u}_i, \tilde{u}_j) \in 4W(k)$ for all $i,\, j \in \{1, \ldots, n\}$, from the proof of Proposition 3.4 (a) (applied with $S = W(k)$ and $q \geq 2$) we get that there exists a $W(k)$-basis $\{u_1, v_1, \ldots, u_n, v_n\}$ for $N_y$ with respect to which the matrix of $b_{N_y}$ is $J(2n)$ and moreover for $i \in \{1, \ldots, n\}$ we have $u_i - \tilde{u}_i \in 2M_y$. Let $F^1_0$ and $F^0_0$ be the $W(k)$-spans of $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ (respectively). The direct sum decomposition $N_y = F^1_0 \oplus F^0_0$ of $W(k)$-modules is such that $F^1_0$ and $\tilde{F}^1_0$ are congruent modulo $2W(k)$ and we have $b_{N_y}(F^1_0, F^1_0) = b_{N_y}(F^0_0, F^0_0) = 0$.

We refer to (7). As $M_y = N_y^s$, we can take the decomposition $M_y = F^1 \oplus F^0$ such that we have $F^1 := (F^1_0)^s$ and $F^0 := (F^0_0)^s$. Thus we have:

**(i)** *the image of the cocharacter $\mu_y : \mathbf{G}_{mW(k)} \to \mathbf{GL}_{M_y}$ that acts on $F^1$ and $F^0$ as desired, fixes all endomorphisms of $M_y$ defined by elements of $\mathcal{B}^{\mathrm{opp}}$.*

As $b_{N_y}(F^1_0, F^1_0) = b_{N_y}(F^0_0, F^0_0) = 0$, we get also that:

**(ii)** *the group scheme $\mathbf{G}_{mW(k)}$ acts via $\mu_y$ on the $W(k)$-span of $b_{N_y}$ and thus also on the $W(k)$-span of $\lambda_{M_y}$, through the inverse of the identity character of $\mathbf{G}_{mW(k)}$.*

From properties (i) and (ii) we get that the cocharacter $\mu_y$ factors through the subgroup scheme of $\mathbf{GSp}(M_y, \lambda_{M_y})$ that fixes the $\mathbf{Z}_{(2)}$-algebra $\mathcal{B}^{\mathrm{opp}}$ of $\mathrm{End}(M_y)$. Therefore the cocharacter $\mu_y$ factors through $\tilde{G}_{W(k)}$. $\square$

**6.1.1. Corollary.** *The normalizer $\tilde{P}_k$ of $\bar{F}_y^1 = F^1 \otimes_{W(k)} k$ in $\tilde{G}_k := \tilde{G}_{W(k)} \times_{W(k)} k$ is a parabolic subgroup of $\tilde{G}_k$ such that we have* $\dim(\tilde{G}_k/\tilde{P}_k) = \frac{n(n-1)}{2} = d$.

*Proof:* Let $q_{N_y}$ be as in Subsection 5.2 and let $q_{\bar{N}_y}$ be the reduction of $q_{N_y}$ modulo $2W(k)$. For $x \in F_0^1 \cup F_0^0$, we have $q_{N_y}(x) = 0$. As we can identify $\tilde{G}_{W(k)}$ with $\mathbf{GSO}(N_y, q_{N_y})$ and as $F^1 = (F_0^1)^s$, we can also identify $\tilde{P}_k$ with the normalizer of $F_0^1/2F_0^1$ in $\mathbf{GSO}(\bar{N}_y, q_{\bar{N}_y})$. Thus the Corollary follows from Lemma 3.5.5. $\square$

**6.2. Good choice of $\mu_y$.** Let $(D_y, \lambda_{D_y}, \mathcal{B})$ be the principally quasi-polarized 2-divisible group over $k$ endowed with endomorphisms of $y^*((\mathcal{A}, \Lambda, \mathcal{B})_{\mathcal{N}_{W(k)}})$. We check that we can choose the cocharacter $\mu_y$ of Proposition 6.1 such that there exists a lift $(\tilde{D}, \lambda_{\tilde{D}}, \mathcal{B})$ of $(D_y, \lambda_{D_y}, \mathcal{B})$ to $W(k)$ with the property that $F^1$ is the Hodge filtration of $M_y$ defined by $\tilde{D}$. We consider the direct sum decomposition $M_y = M_{y(0)} \oplus M_{y(1)} \oplus M_{y(2)}$ left invariant by $\Phi_y$ and such that all slopes of $(M_{y(l)}, \Phi_y)$ are 0 if $l = 0$, are 1 if $l = 2$, and belong to the interval $(0, 1)$ if $l = 1$. To it corresponds a product decomposition $D_y = \prod_{l=0}^2 D_{y(l)}$.

We consider the Newton type of cocharacter $\nu_y : \mathbf{G}_{m\,W(k)} \to \mathbf{GL}_{M_y}$ that acts on $M_{y(l)}$ via the $l$-th power of the identity character of $\mathbf{G}_{m\,W(k)}$. The endomorphisms of $M_y$ defined by elements of $\mathcal{B}^{\mathrm{opp}}$ are fixed by (i.e., commute with) $\Phi_y$ and moreover we have $\lambda_{M_y}(\Phi_y(u), \Phi_y(v)) = 2\sigma(\lambda_{M_y}(u, v))$, for all $u$, $v \in M_y$. This implies that the cocharacter $\nu_y$ fixes all these endomorphisms of $M_y$ and normalizes the $W(k)$-span of $\lambda_{M_y}$. Thus $\nu_y$ factors through $\tilde{G}_{W(k)}$. The special fibre of $\nu_y$ normalizes $\bar{F}_y^1 = F^1/2F^1$ (i.e., the kernel of $\Phi_y$ modulo $2W(k)$) and therefore it factors through $\tilde{P}_k$. We can replace the role of $\mu_y$ by the one of an inner conjugate of it through an arbitrary element $\tilde{g} \in \tilde{G}_{W(k)}(W(k))$ that lifts an element of $\tilde{P}_k(k)$. But there exist such elements $\tilde{g}$ with the property that $\tilde{g}\mu_y\tilde{g}^{-1}$ and $\nu_y$ commute. Thus not to introduce extra notations, we will assume that $\mu_y$ and $\nu_y$ commute. Therefore we have a direct sum decomposition $F^1 = \oplus_{l=0}^2 F_{(l)}^1$, where $F_{(l)}^1 := F^1 \cap M_{y(l)}$. There exists a unique 2-divisible group $\tilde{D}_{(l)}$ over $W(k)$ that lifts $D_{y(l)}$ and whose Hodge filtration is $F_{(l)}^1$, cf. [16, 1.6 (ii) of p. 186] applied to the Honda triple $(M_{y(l)}, \frac{1}{2}\Phi_y(F_{(l)}^1), \Phi_y)$. Let $\tilde{D} := \prod_{l=0}^2 \tilde{D}_{(l)}$; the fact that there exists a lift $(\tilde{D}, \lambda_{\tilde{D}}, \mathcal{B}^{\mathrm{opp}})$ of $(D_y, \lambda_{D_y}, \mathcal{B})$ to $W(k)$ is also implied by loc. cit.

**6.2.1. Remark.** Let $(\tilde{A}, \lambda_{\tilde{A}}, \mathcal{B})$ be the principally polarized abelian scheme over $W(k)$ endowed with endomorphisms that lifts $(A, \lambda_A, \mathcal{B})$ and whose principally quasi-polarized 2-divisible group endowed with endomorphisms is $(\tilde{D}, \lambda_{\tilde{D}}, \mathcal{B})$. Let $\mathrm{Spec}(W(k)) \to \mathcal{M}_{W(k)}$ be the morphism of $W(k)$-schemes associated to $(\tilde{A}, \lambda_{\tilde{A}})$ and its symplectic similitude structures that lift those of $(A, \lambda_A)$. As we are in the case (D), we have $G \neq G_1$ and thus the Hasse principle fails for $G$. Therefore we can not prove directly that this morphism of $W(k)$-schemes factors through $\mathcal{N}_{W(k)}$. This explains why in the below Subsections 6.5 and 6.6 we will work mainly modulo 2 and not directly in mixed characteristic $(0, 2)$.

**6.3. Defining $\tilde{U}$.** We have a natural direct sum decomposition of $W(k)$-modules $\mathrm{End}(M_y) = \mathrm{End}(F^1) \oplus \mathrm{End}(F^0) \oplus \mathrm{Hom}(F^1, F^0) \oplus \mathrm{Hom}(F^0, F^1)$. Let $\tilde{U}$ be the flat, closed subgroup scheme of $\mathbf{GL}_{M_y}$ defined by the rule: if $\tilde{R}$ is a commutative $W(k)$-algebra, then

$$\tilde{U}(\tilde{R}) = 1_{M_y \otimes_{W(k)} \tilde{R}} + (\mathrm{Lie}(\tilde{G}_{W(k)}) \cap \mathrm{Hom}(F^1, F^0)) \otimes_{W(k)} \tilde{R}$$

(the last intersection being taken inside $\mathrm{End}(M_y)$). The group scheme $\tilde{U}$ is smooth, connected, commutative, and its Lie algebra is the direct summand $\mathrm{Lie}(\tilde{G}_{W(k)}) \cap \mathrm{Hom}(F^1, F^0)$ of $\mathrm{Hom}(F^1, F^0) = \mathrm{Hom}(F^1, M_y/F^1)$. Thus $\mathrm{Lie}(\tilde{U}) \subseteq \mathrm{Lie}(\tilde{G}_{W(k)})$. This implies $\tilde{U}_{B(k)} \leqslant \tilde{G}_{B(k)}$, cf. [6, Ch. II, Subsection 7.1]. Thus $\tilde{U} \leqslant \tilde{G}_{W(k)}$. The group scheme $\tilde{U}$ acts trivially on both $F^0$ and $M_y/F^0$ and the natural morphism $\tilde{U}_k \to \tilde{G}_k/\tilde{P}_k$ of $k$-schemes is an open embedding. Thus the relative dimension of $\tilde{U}$ over $W(k)$ is $\dim(\tilde{G}_k/\tilde{P}_k)$ and therefore (cf. Corollary 6.1.1) it is $d$.

Let $\mathrm{Spf}(R(d))$ be the completion of $\tilde{U}$ along its identity section. We can identify $R(d) = W(k)[[t_1, \ldots, t_d]]$ in such a way that the ideal $(t_1, \ldots, t_d)$ defines the identity section of $\tilde{U}$. Let $\Phi_{R(d)}$ be the Frobenius lift of $R(d)$ that is compatible with $\sigma$ and takes $t_i$ to $t_i^2$ for $i \in \{1, \ldots, d\}$. Let $d\Phi_{R(d)}$ be the differential map of $\Phi_{R(d)}$. Let

$$\tilde{u}_{\mathrm{univ}} \in \tilde{U}(R(d))$$

be the universal element defined by the natural morphism $\mathrm{Spec}(R(d)) \to \tilde{U}$ of $W(k)$-schemes.

The existence of $\tilde{D}$ allows us to apply [14, Thm. 10]: from loc. cit. we get the existence of a unique connection $\nabla_y : M_y \otimes_{W(k)} R(d) \to M_y \otimes_{W(k)} \oplus_{i=1}^d R(d) dt_i$ such that we have an identity

$$\nabla_y \circ \tilde{u}_{\mathrm{univ}}(\Phi_y \otimes \Phi_{R(d)}) = (\tilde{u}_{\mathrm{univ}}(\Phi_y \otimes \Phi_{R(d)}) \otimes d\Phi_{R(d)}) \circ \nabla_y \tag{13}$$

of maps from $M_y \otimes_{W(k)} R(d)$ to $M_y \otimes_{W(k)} \oplus_{i=1}^d R(d) dt_i$; the connection $\nabla_y$ is integrable and topologically nilpotent. Let $\nabla_{\mathrm{triv}}$ be the flat connection on $M_y \otimes_{W(k)} R(d)$ that annihilates $M_y \otimes 1$. Due to Formula (13), the connections $\nabla_{\mathrm{triv}} - \tilde{u}_{\mathrm{univ}}^{-1} d\tilde{u}_{\mathrm{univ}}$ and $\nabla_y$ are congruent modulo $(t_1, \ldots, t_d)^2$. From this and the very definition of $\tilde{u}_{\mathrm{univ}}$ we get:

**6.3.1. Fact.** *The Kodaira–Spencer map of $\nabla_y$ is injective and its image is a direct summand $\mathfrak{I}_y$ of* $\mathrm{Hom}(F^1, M_y/F^1) \otimes_{W(k)} R(d)$.

**6.4. A deformation.** We recall that the categories of 2-divisible groups over $\mathrm{Spf}(R(d)/2R(d))$ and respectively over $\mathrm{Spec}(R(d)/2R(d))$, are canonically isomorphic (cf. [11, Lemma 2.4.4]). From this and [14, Thm. 10] we get the existence of a 2-divisible group $\tilde{D}_{R(d)/2R(d)}$ over $R(d)/2R(d)$ whose $F$-crystal over $R(d)/2R(d)$ is $(M_y \otimes_{W(k)} R(d), \tilde{u}_{\mathrm{univ}}(\Phi_y \otimes \Phi_{R(d)}), \nabla_y)$. We have

$$\tilde{u}_{\mathrm{univ}}(\Phi_y \otimes \Phi_{R(d)}) \circ b = b \circ \tilde{u}_{\mathrm{univ}}(\Phi_y \otimes \Phi_{R(d)}) \ \ \forall b \in \mathcal{B}^{\mathrm{opp}}. \tag{14}$$

As the connection $\nabla_y$ is uniquely determined by $\tilde{u}_{\mathrm{univ}}(\Phi_y \otimes \Phi_{R(d)})$ and due to Formula (14), the connection $\nabla_y$ is invariant under the natural action of the group of invertible elements of $\mathcal{B}^{\mathrm{opp}}$ on $M_y \otimes_{W(k)} R(d)$. But each element of $\mathcal{B}^{\mathrm{opp}}$ is a sum of two invertible elements of $\mathcal{B}^{\mathrm{opp}}$. Thus as the ring $R(d)/2R(d)$ has a finite 2-basis, from the fully faithfulness part of [5, Thm. 4.1.1 or 4.2.1] we get first that the 2-divisible group $\tilde{D}_{R(d)/2R(d)}$ is uniquely determined (by its $F$-crystal over $R(d)/2R(d)$) and second that each $b \in \mathcal{B}^{\mathrm{opp}}$ is the crystalline realization of a unique endomorphism $b$ of $\tilde{D}_{R(d)/2R(d)}$. Thus we have a natural $\mathbf{Z}_{(2)}$-monomorphism $\mathcal{B} \hookrightarrow \mathrm{End}(\tilde{D}_{R(d)/2R(d)})$. A similar argument shows that there exists a unique principal quasi-polarization $\lambda_{\tilde{D}_{R(d)/2R(d)}}$ of $\tilde{D}_{R(d)/2R(d)}$ such that $\mathbf{D}(\lambda_{\tilde{D}_{R(d)/2R(d)}})$ is defined by the alternating form $\lambda_{M_y}$ on $M_y \otimes_{W(k)} R(d)$. As $\mathrm{Lie}(\tilde{G}_{W(k)})$ is the direct summand of $\mathrm{Lie}(\mathbf{GSp}(M_y, \lambda_{M_y}))$ that annihilates $\mathrm{Im}(\mathcal{B}^{\mathrm{opp}} \to \mathrm{End}(M_y))$ and as $\nabla_y$ annihilates $\mathrm{Im}(\mathcal{B}^{\mathrm{opp}} \to \mathrm{End}(M_y) \otimes_{W(k)} R(d))$, we get that

$$\nabla_y - \nabla_{\mathrm{triv}} \in \mathrm{Lie}(\tilde{G}_{W(k)}) \otimes_{W(k)} \oplus_{i=1}^d R(d)dt_i \subseteq \mathrm{End}(M_y) \otimes_{W(k)} \oplus_{i=1}^d R(d)dt_i.$$

This implies that the direct summand $\mathfrak{I}_y$ of $\mathrm{Hom}(F^1, M_y/F^1) \otimes_{W(k)} R(d)$ introduced in Fact 6.3.1, is $\mathrm{Lie}(\tilde{U}) \otimes_{W(k)} R(d)$. Let

$$\mathcal{C}_y := (M_y \otimes_{W(k)} R(d), F^1 \otimes_{W(k)} R(d), \tilde{u}_{\mathrm{univ}}(\Phi_y \otimes \Phi_{R(d)}), \nabla_y, \lambda_{M_y}, \mathcal{B}^{\mathrm{opp}}).$$

From Serre–Tate deformation theory we get the existence of a unique triple

$$(A'_{R(d)/2R(d)}, \lambda_{A'_{R(d)/2R(d)}}, \mathcal{B})$$

over $R(d)/2R(d)$ that lifts $(A, \lambda_A, \mathcal{B})$ and whose principally quasi-polarized 2-divisible group over $R(d)/2R(d)$ endowed with endomorphisms is $(\tilde{D}_{R(d)/2R(d)}, \lambda_{\tilde{D}_{R(d)/2R(d)}}, \mathcal{B})$. To $(A'_{R(d)/2R(d)}, \lambda_{A'_{R(d)/2R(d)}})$ and its symplectic similitude structures that lift those of $(A, \lambda_A)$, corresponds a morphism

$$m_{R(d)/2R(d)} : \mathrm{Spec}(R(d)/2R(d)) \to \mathcal{M}_k$$

of $k$-schemes.

**6.5. Specializing to $y$.** Let $R = W(k)[[t]]$ be as in Subsection 2.1. Let

$$n_{R/2R} : \mathrm{Spec}(R/2R) \to \mathcal{N}_k$$

be an arbitrary morphism of $k$-schemes that lifts the point $y \in \mathcal{N}_{W(k)}(k) = \mathcal{N}_k(k)$. Let $m_{R/2R} : \mathrm{Spec}(R/2R) \to \mathcal{M}_k$ be the composite of $n_{R/2R}$ with the morphism $\mathcal{N}_k \to \mathcal{M}_k$. Let $k_1$ be an algebraic closure of $k((t))$. Let the morphism $y_1 : \mathrm{Spec}(k_1) \to \mathcal{N}_{W(k_1)}$ be defined naturally by $n_{R/2R}$. We recall from Subsection 2.1 that $U_m = k[[t]]/(t^m)$, where $m \in \mathbf{N}$. Let

$$n_{U_m} : \mathrm{Spec}(U_m) \to \mathcal{N}_k \text{ and } m_{U_m} : \mathrm{Spec}(U_m) \to \mathcal{M}_k$$

be the composites of the closed embedding $\mathrm{Spec}(U_m) \hookrightarrow \mathrm{Spec}(R/2R)$ with $n_{R/2R}$ and $m_{R/2R}$ (respectively). Let $(M_R, \Phi_{M_R}, \nabla_{M_R}, \lambda_{M_R})$ be the principally quasi-polarized $F$-crystal over $R/2R$ of the pull back $(\tilde{A}_{R/2R}, \lambda_{\tilde{A}_{R/2R}})$ through $m_{R/2R}$ of the universal principally polarized abelian scheme over $\mathcal{M}_k$; thus $(M_R, \lambda_{M_R})$ is a symplectic space over $R$ of rank $\dim_{\mathbf{Q}}(W)$, etc. We have a natural $\mathbf{Z}_2$-monomorphism $\mathcal{B}^{\mathrm{opp}} \hookrightarrow \mathrm{End}(M_R, \Phi_{M_R}, \nabla_{M_R})$. Let $\mathcal{K}$ be the field of fractions of $R$. Let $F^1_{R/2R}$ be the Hodge filtration of $M_R/2M_R$ defined by $\tilde{A}_{R/2R}$ (i.e., the kernel of the reduction of $\Phi_{M_R}$ modulo $2R$). Let $\tilde{G}'_R$ be the Zariski closure in $\mathbf{GL}_{M_R}$ of the identity component of the subgroup of $\mathbf{GSp}(M_R, \lambda_{M_R})_{\mathcal{K}}$ that fixes the $\mathcal{K}$-subalgebra $\mathcal{B}^{\mathrm{opp}} \otimes_{\mathbf{Z}_{(2)}} \mathcal{K}$ of $\mathrm{End}(M_R \otimes_R \mathcal{K})$. Let $M_R = N_R^s$ be the decomposition in $R$-modules that corresponds naturally to (4). Let $b_{N_R}$ be the perfect bilinear form on $N_R$ that corresponds naturally to $b_{\mathcal{V}}$ of Subsubsection 3.5.3. By applying Theorem 5.1 to $y_1$, we get that $b_{N_R}$ modulo $2R$ is alternating. Thus the formula $q_{N_R}(x) := \frac{b_{N_R}(x,x)}{2}$ defines a quadratic form on $N_R$. We can redefine $\tilde{G}'_R$ as $\mathbf{GSO}(N_R, q_{N_R})$ and therefore $\tilde{G}'_R$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{M_R}$, cf. Proposition 3.4 (b).

Let $\tilde{P}'_{k((t))}$ be the parabolic subgroup of $\tilde{G}'_{k((t))}$ that is the normalizer of $F^1_{R/2R} \otimes_{R/2R} k((t))$ in $\tilde{G}'_{k((t))}$, cf. Corollary 6.1.1 applied to $y_1$. The $R$-scheme of parabolic subgroup schemes of $\tilde{G}'_R$ is projective, cf. [12, Vol. III, Exp. XXVI, Cor. 3.5]. Therefore the Zariski closure $\tilde{P}'_{R/2R}$ of $\tilde{P}'_{k((t))}$ in $\tilde{G}'_{R/2R}$ is a parabolic subgroup scheme of $\tilde{G}'_{R/2R}$ that normalizes $F^1_{R/2R}$ and that (cf. Corollary 6.1.1 applied to $y_1$) has the same relative dimension as $\tilde{P}_k$. Thus we have a monomorphism $\tilde{P}'_k \hookrightarrow \tilde{P}_k$ over $k$ which by reasons of dimensions is an isomorphism over $k$, to be viewed as an identification. The special fibre $\bar{\mu}_y$ of $\mu_y$ factors through $\tilde{P}_k = \tilde{P}'_k$. From [12, Vol. II, Exp. IX, Thms. 3.6 and 7.1] we get that $\bar{\mu}_y$ lifts to a cocharacter $\mu_{R/2R} : \mathbf{G}_{m\,R/2R} \to \tilde{P}'_{R/2R}$ which at its turn lifts to a cocharacter $\mu_R : \mathbf{G}_{m\,R} \to \tilde{G}'_R$. Let $M_R = F^{1\prime}_R \oplus F^{0\prime}_R$ be the direct sum decomposition defined naturally by $\mu_R$. From constructions we get that

$$F^{1\prime}_R / 2F^{1\prime}_R = F^1_{R/2R} \ \text{ and } \ F^{0\prime}_R \otimes_R k = F^0/2F^0.$$

**6.6. Theorem.** *For $m \in \mathbf{N}$ let $\mathcal{T}_m := n^*_{U_m}((\mathcal{A}, \Lambda, \mathcal{B})_{\mathcal{N}_k})$. There exists a morphism $u_m : \mathrm{Spec}(U_m) \to \mathrm{Spec}(R(d)/2R(d))$ of $k$-schemes such that $u_m^*(A'_{R(d)/2R(d)}, \lambda_{A'_{R(d)/2R(d)}}, \mathcal{B})$ is isomorphic to $\mathcal{T}_m$, under an isomorphism that lifts the identity automorphism of $(A, \lambda_A, \mathcal{B})$.*

*Proof:* Let

$$(M_R(m), F^{1\prime}_R(m), \Phi_{M_R}(m), \nabla_{M_R}(m), \lambda_{M_R}(m), \tilde{G}'_R(m))$$

be the reduction of $(M_R, F^{1\prime}_R, \Phi_{M_R}, \nabla_{M_R}, \lambda_{M_R}, \tilde{G}'_R)$ modulo the ideal $(t^m)$ of $R$. This Theorem is a geometric variant of a slight modification of [14, Thm. 10 and Rm. iii)]. Following loc. cit., we show by induction on $m \in \mathbf{N}$ that the morphism $u_m$ exists and that there exists a $W(k)$-homomorphism $h(m) : R(d) \to R/(t^m)$ which has the following two properties:

**(i)** it maps $(t_1, \ldots, t_d)$ to $(t)/(t^m)$ and modulo $2W(k)$ it defines the morphism $u_m$;

**(ii)** the extension of $\mathcal{C}_y$ via $h(m)$ is isomorphic to $(M_R(m), F_R^{1\prime}(m), \Phi_{M_R}(m), \nabla_{M_R}(m), \lambda_{M_R}(m), \mathcal{B}^{\text{opp}})$ under an isomorphism $\mathcal{Y}(m)$ that modulo $(t)/(t^m)$ is $1_{M_y}$.

The case $m = 1$ follows from constructions. The passage from $m$ to $m+1$ goes as follows. We endow the ideal $J_m := (t^m)/(t^{m+1})$ of $R/(t^{m+1})$ with the natural divided power structure; thus $J_m^{[2]} = 0$. The ideal $(2, J_m)$ of $R/(t^{m+1})$ has a natural divided power structure that extends the one of $J_m$. Let $\tilde{h}(m+1) : R(d) \to R/(t^{m+1})$ be a $W(k)$-homomorphism that lifts $h(m)$. Let $\tilde{u}_{m+1} : \text{Spec}(U_{m+1}) \to \text{Spec}(R(d)/2R(d))$ be defined by $\tilde{h}(m+1)$ modulo $2W(k)$. We apply the crystalline Dieudonné functor $\mathbf{D}$ in the context:

– of the principally quasi-polarized 2-divisible groups endowed with endomorphisms of $\tilde{u}_{m+1}^*(A'_{R(d)/2R(d)}, \lambda_{A'_{R(d)/2R(d)}}, \mathcal{B})$ and of $\mathcal{T}_{m+1}$, and

– of the thickenings attached naturally to the closed embeddings $\text{Spec}(U_m) \hookrightarrow \text{Spec}(R/(t^{m+1}))$ and $\text{Spec}(U_{m+1}) \hookrightarrow \text{Spec}(R/(t^{m+1}))$.

We get that the extension of $\mathcal{C}_y$ through $\tilde{h}(m+1)$ is isomorphic to the sextuple $(M_R(m+1), \tilde{F}_R^1(m+1), \Phi_{M_R}(m+1), \nabla_{M_R}(m+1), \lambda_{M_R}(m+1), \mathcal{B}^{\text{opp}})$ under an isomorphism to be denoted by $\mathcal{Y}(m+1)$. As $\Phi_{R(d)}(t_1, \ldots, t_d) \subseteq (t_1, \ldots, t_d)^2$, the isomorphism $\mathcal{Y}(m)$ is uniquely determined by the property (ii). Thus $\mathcal{Y}(m+1)$ lifts $\mathcal{Y}(m)$ and therefore $\tilde{F}_R^1(m+1)$ is a direct summand of $M_R(m+1)$ that lifts $F_R^{1\prime}(m)$.

We check that under $\mathcal{Y}(m+1)$, the reductive subgroup scheme $\tilde{G}_{W(k)} \times_{W(k)} R(d)$ of $\mathbf{GL}_{M_y \otimes_{W(k)} R(d)}$ pulls back to the reductive subgroup scheme $\tilde{G}'_R(m+1)$ of $\mathbf{GL}_{M_R(m+1)}$. It suffices to check that under the isomorphism $\mathcal{Y}(m+1)[\frac{1}{2}]$, the reductive subgroup scheme $\tilde{G}_{W(k)} \times_{W(k)} R(d)[\frac{1}{2}]$ of $\mathbf{GL}_{M_y \otimes_{W(k)} R(d)[\frac{1}{2}]}$ pulls back to the reductive subgroup scheme $\tilde{G}'_R(m+1)[\frac{1}{2}]$ of $\mathbf{GL}_{M_R(m+1)[\frac{1}{2}]}$. But this holds as $\tilde{G}_{W(k)} \times_{W(k)} R(d)[\frac{1}{2}]$ (resp. as $\tilde{G}'_R \times_R R[\frac{1}{2}]$) is the identity component of the subgroup scheme of $\mathbf{GSp}(M_y \otimes_{W(k)} R(d)[\frac{1}{2}], \lambda_{M_y})$ (resp. of $\mathbf{GSp}(M_R[\frac{1}{2}], \lambda_{M_R})$) that fixes all elements of $\mathcal{B}^{\text{opp}}$. For $\tilde{m} \in \{m, m+1\}$ we will identify naturally $\tilde{U} \times_{W(k)} R/(t^{\tilde{m}})$ with a closed subgroup scheme of $\tilde{G}'_R(m)$ and thus we will view $\tilde{U}(R/(t^{\tilde{m}}))$ as a subgroup of $\tilde{G}'_R(R/(t^{\tilde{m}})) = \tilde{G}'_R(\tilde{m})(R/(t^{\tilde{m}}))$.

Let $\mu_{R,m+1} : \mathbf{G}_{m R/(t^{m+1})} \to \tilde{G}'_R(m+1)$ be the reduction of the cocharacter $\mu_R$ modulo $(t^{m+1})$. Let $\mu_{y,m+1} : \mathbf{G}_{m R/(t^{m+1})} \to \tilde{G}'_R(m+1)$ be the cocharacter obtained naturally from $\mu_{y\,R(d)}$ via $\tilde{h}(m+1)$ and $\mathcal{Y}(m+1)$. As over $W(k)$ the two cocharacters $\mu_{R,m+1}$ and $\mu_{y,m+1}$ of $\tilde{G}'_R(m+1)$ coincide, there exists an element $h_3 \in \text{Ker}(\tilde{G}'_R(R/(t^{m+1})) \to \tilde{G}'_R(R/(t)))$ such that we have an identity

$$h_3 \mu_{R,m+1} h_3^{-1} = \mu_{y,m+1}$$

(cf. [12, Vol. II, Exp. IX, Thm. 3.6]). As $\tilde{F}_R^1(m+1)$ lifts $F_R^{1\prime}(m)$, we can write $h_3 = h_1 h_2$, where $h_1 \in \text{Ker}(\tilde{U}(R/(t^{m+1})) \to \tilde{U}(R/(t^m)))$ and where $h_2 \in \text{Ker}(\tilde{G}'_R(R/(t^{m+1})) \to \tilde{G}'_R(R/(t)))$ normalizes $F_R^{1\prime}(m+1)$. We have $h_1(F_R^{1\prime}(m+1)) = h_3(F_R^{1\prime}(m+1)) = \tilde{F}_R^1(m+1)$.

As $h_1 \in \mathrm{Ker}(\tilde{U}(R/(t^{m+1})) \to \tilde{U}(R/(t^m)))$ and (see Subsection 6.4) as we have an identity $\mathfrak{I}_y = \mathrm{Lie}(\tilde{U}) \otimes_{W(k)} R(d) \subseteq \mathrm{Hom}(F^1, M_y/F^1) \otimes_{W(k)} R(d)$, we can replace $\tilde{h}(m+1)$ by another lift $h(m+1)$ of $h(m)$ such that its corresponding $h_1$ element is the identity. Let $u_{m+1} : \mathrm{Spec}(U_{m+1}) \to \mathrm{Spec}(R(d)/2R(d))$ be defined by $h(m+1)$ modulo $2W(k)$.

Thus we have $F_R^{1\prime}(m+1) == \tilde{F}_R^1(m+1)$. The fact that $u_{m+1}^*(A'_{R(d)/2R(d)}, \lambda_{A'_{R(d)/2R(d)}}, \mathcal{B})$ is $\mathcal{T}(m+1)$ follows from the deformation theories of Subsubsection 1.2.3 applied in the context of the nilpotent thickening attached naturally to the closed embedding $\mathrm{Spec}(U_m) \hookrightarrow \mathrm{Spec}(U_{m+1})$. This ends the induction. $\square$

From Theorem 6.6 we get directly that for each $m \in \mathbf{N}$, the point $m_{U_m} \in \mathcal{M}_k(U_m)$ is $m_{R(d)/2R(d)} \circ u_m \in \mathcal{M}_k(U_m)$. This implies that:

**6.6.1. Corollary.** *The morphism* $m_{R/2R} : \mathrm{Spec}(R/2R) \to \mathcal{M}_k$ *factors through* $m_{R(d)/2R(d)} : \mathrm{Spec}(R(d)/2R(d)) \to \mathcal{M}_k$.

**6.7. End of the proof of 1.4 (b).** Let $\bar{\mathcal{O}}_y$ (resp. $\bar{\mathcal{O}}_y^{\mathrm{big}}$) be the completion of the local ring of the $k$-valued point of $\mathcal{N}_{k\mathrm{red}} := \mathcal{N}_{k(v)\mathrm{red}} \times_{k(v)} k$ (resp. of $\mathcal{M}_k$) defined by $y$. The scheme $\bar{\mathcal{O}}_y$ is reduced (as the excellent scheme $\mathcal{N}_{k\mathrm{red}}/H_0$ is so). Thus the normalization $\bar{\mathcal{O}}_y^{\mathrm{n}}$ of $\bar{\mathcal{O}}_y$ is well defined. The complete, excellent $k$-algebra $\bar{\mathcal{O}}_y$ has dimension $d$.

Due to Fact 6.3.1, the $k$-homomorphism $\bar{m}_y : \bar{\mathcal{O}}_y^{\mathrm{big}} \to R(d)/2R(d)$ defined naturally by $m_{R(d)/2R(d)}$ is onto. From Corollary 6.6.1 applied to morphisms $n_{R/2R} : \mathrm{Spec}(R/2R) \to \mathcal{N}_k$ of $k$-schemes that factor through $\mathrm{Spec}(\bar{\mathcal{O}}_y^{\mathrm{n}})$, we get that the natural finite $k$-homomorphism $\bar{\mathcal{O}}_y^{\mathrm{big}} \to \bar{\mathcal{O}}_y^{\mathrm{n}}$ factors through $\bar{m}_y$ giving birth to a finite $\bar{\mathcal{O}}_y^{\mathrm{big}}$-homomorphism $\bar{q}_y : R(d)/2R(d) \to \bar{\mathcal{O}}_y^{\mathrm{n}}$. Thus we have a commutative diagram

$$\begin{array}{ccc} \bar{\mathcal{O}}_y^{\mathrm{big}} & \xrightarrow{\bar{e}_y} & \bar{\mathcal{O}}_y \\ \Big\downarrow{\scriptstyle \bar{m}_y} & & \Big\downarrow{\scriptstyle \bar{i}_y} \\ R(d)/2R(d) & \xrightarrow{\bar{q}_y} & \bar{\mathcal{O}}_y^{\mathrm{n}}, \end{array}$$

where $\bar{e}_y$ is the natural $k$-epimorphism and $\bar{i}_y$ is the natural inclusion. By reasons of dimensions, the $k$-homomorphism $\bar{q}_y$ is injective. Therefore we have $\mathrm{Ker}(\bar{m}_y) = \mathrm{Ker}(\bar{e}_y)$. Thus $\bar{q}_y$ and $\bar{i}_y$ are both $k$-isomorphisms. This implies that $\mathcal{N}_{k\mathrm{red}}$ is regular and formally smooth over $k$ at the $k$-valued point defined by $y$ and that the morphism $m_{R(d)/2R(d)} : \mathrm{Spec}(R(d)/2R(d)) \to \mathcal{M}_k$ of $k$-schemes factors through a formally étale morphism

$$n_{R(d)/2R(d)} : \mathrm{Spec}(R(d)/2R(d)) \to \mathcal{N}_{k\mathrm{red}}$$

of $k$-schemes. This implies that the $k(v)$-scheme $\mathcal{N}_{k(v)\mathrm{red}}$ is regular and formally smooth. Thus Theorem 1.4 (b) holds.

**6.8. Corollary.** *The reduced* $k(v)$*-scheme of* $\mathcal{N}_{k(v)}^{\mathrm{n}}$ *is regular and formally smooth.*

*Proof:* Both morphisms $n_{R(d)/2R(d)} : \mathrm{Spec}(R(d)/2R(d)) \to \mathcal{N}_{k\mathrm{red}}$ and $n_{R/2R} : \mathrm{Spec}(R/2R) \to \mathcal{N}_k$ of $k$-schemes (we introduced in Subsections 6.7 and respectively 6.5) factor through

the reduced scheme of $\mathcal{N}^{\mathrm{n}}_{k}$. Thus as in Subsection 6.7 we argue that the reduced $k(v)$-scheme of $\mathcal{N}^{\mathrm{n}}_{k(v)}$ is regular and formally smooth. $\square$

## 7. Proof of 1.4 (c)

The following Proposition is a particular case of [34, Cor. 3.8].

**7.1. Proposition.** *The ordinary locus of $\mathcal{N}^{\mathrm{n}}_{k(v)} \setminus \mathcal{N}^{\mathrm{s}}_{k(v)}$ is empty (i.e., if the abelian variety $A$ is ordinary, then the point $y : \mathrm{Spec}(k) \to \mathcal{N}_{W(k)}$ does not factor through $\mathcal{N}^{\mathrm{n}}_{k(v)} \setminus \mathcal{N}^{\mathrm{s}}_{k(v)}$).*

**7.2. Proposition.** *The formally smooth locus $\mathcal{N}^{\mathrm{s}}$ is an open subscheme of $\mathcal{N}^{\mathrm{n}}$ with the properties that $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})} = \mathcal{N}^{\mathrm{n}}_{E(G,\mathcal{X})} = \mathrm{Sh}(G,\mathcal{X})/H_2$ and that its special fibre $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is an open closed subscheme of $\mathcal{N}^{\mathrm{n}}_{k(v)}$.*

*Proof:* Based on Subsection 4.1.1, we only have to show that $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is a closed subscheme of $\mathcal{N}^{\mathrm{n}}_{k(v)}$. To check this, we can assume that $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is non-empty. The connected components of $\mathcal{N}^{\mathrm{s}}_{k}$ and $\mathcal{N}^{\mathrm{n}}_{k}$ are irreducible (cf. Corollary 6.8 for $\mathcal{N}^{\mathrm{n}}_{k}$). Let $\mathcal{N}^{\mathrm{s}0}_{W(k)}$ and $\mathcal{N}^{\mathrm{n}0}_{W(k)}$ be open subschemes of $\mathcal{N}^{\mathrm{s}}_{W(k)}$ and $\mathcal{N}^{\mathrm{n}}_{W(k)}$ (respectively) whose special fibres have non-empty connected components permuted transitively by $H_0$ and such that we have a quasi-finite monomorphism $\mathcal{N}^{\mathrm{s}0}_{W(k)}/H_0 \hookrightarrow \mathcal{N}^{\mathrm{n}0}_{W(k)}/H_0$ between integral $W(k)$-schemes. The special fibres $\mathcal{N}^{\mathrm{s}0}_{k}/H_0$ and $\mathcal{N}^{\mathrm{n}0}_{k}/H_0$ of $\mathcal{N}^{\mathrm{s}0}_{W(k)}/H_0$ and $\mathcal{N}^{\mathrm{n}0}_{W(k)}/H_0$ (respectively) are irreducible and generically smooth. Let $\mathrm{Spec}(R_0)$ be an affine, open subscheme of $\mathcal{N}^{\mathrm{n}0}_{W(k)}/H_0$ such that $R_0 \neq 2R_0$; it is a normal, integral, excellent, flat $W(k)$-algebra such that $(R_0/2R_0)_{\mathrm{red}}$ is a smooth, integral $k$-algebra (cf. Corollary 6.8) and $R_0/2R_0$ is a generically smooth $k$-algebra. Thus from Hironaka Lemma (see [20, Ch. III, Lemma 9.12]) applied naturally in mixed characteristic $(0,2)$ to $R_0$ and to $2 \in R_0$, we get that $R_0/2R_0$ is an integral and thus a smooth $k$-algebra. This implies that $\mathcal{N}^{\mathrm{n}0}_{W(k)}$ is a formally smooth $W(k)$-scheme and therefore we can choose $\mathcal{N}^{\mathrm{s}0}_{W(k)}$ to be $\mathcal{N}^{\mathrm{n}0}_{W(k)}$. Thus $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is a closed subscheme of $\mathcal{N}^{\mathrm{n}}_{k(v)}$. $\square$

**7.3. Proposition.** *The ordinary locus of $\mathcal{N}_k$ is Zariski dense in $\mathcal{N}_k$.*

*Proof:* Let $y \in \mathcal{N}_{W(k)}(k)$. We will use the notations of Subsection 6.2. Let $(\bar{M}_y, \phi_y, \upsilon_y)$ be the Dieudonné module of $A[2]$; thus $(\bar{M}_y, \phi_y) = (M_y, \Phi_y) \otimes_{W(k)} k$. We consider the inverse $l_y : L^*_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k) \tilde{\to} M_y$ of the dual of the isomorphism $j_y$ of Subsection 5.2. We view naturally $G_{\mathbf{Z}_2}$ as a closed subgroup scheme of $\mathbf{GL}_{L^*_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2}$. As $G_{\mathbf{Z}_2}$ is split, there exists a cocharacter $\mu_{\acute{e}t} : \mathbf{G}_{m\mathbf{Z}_2} \to G_{\mathbf{Z}_2}$ such that the cocharacter

$$l_y \mu_{\acute{e}t\, W(k)} l_y^{-1} : \mathbf{G}_{m W(k)} \to \tilde{G}_{W(k)}$$

is $\tilde{G}_{W(k)}(W(k))$-conjugate to the cocharacter $\mu_y : \mathbf{G}_{m W(k)} \to \tilde{G}'_{W(k)}$ of Proposition 6.1.

By replacing $l_y$ with its left composite with an element of $\tilde{G}_{W(k)}(W(k))$, we can assume that in fact we have $l_y \mu_{\acute{e}t\, W(k)} l_y^{-1} = \mu_y$. Thus $l_y^{-1} \Phi_y l_y$ is a $\sigma$-linear endomorphism

of $L^*_{(2)}\otimes_{\mathbf{Z}_{(2)}} W(k)$ of the form $g_{ét}(1_{L^*_{(2)}}\otimes\sigma)\mu_{étW(k)}(\frac{1}{2})$ for some element $g_{ét}\in G_{\mathbf{Z}_{(2)}}(W(k))$. As $(L^*_{(2)}\otimes_{\mathbf{Z}_{(2)}} W(k), (1_{L^*_{(2)}}\otimes\sigma)\mu_{étW(k)}(\frac{1}{2}))$ is an ordinary $F$-crystal over $k$, there exists $g_y\in \tilde G_{W(k)}(W(k))$ such that $(M_y, g_y\Phi_y)$ is an ordinary $F$-crystal over $k$. Based on this, as in the last paragraph of Subsubsection 5.1.4 we argue that there exist elements $\tilde u\in \tilde U(k)$ such that the Dieudonné module $(\bar M_y, \tilde u\phi_y, \upsilon_y(\tilde u)^{-1})$ is ordinary; the only think we need to add is that the roles played by $h_S:\mathrm{Spec}(S)\to \tilde H$ and $\tilde P\times_k \mathrm{Spec}(S)$ in Subsubsection 5.1.4, are played now by $\tilde U_k\hookrightarrow \tilde G_k$ and $\tilde P_k\times_k \tilde U_k$ (respectively).

As the reduction modulo $2W(k)$ of the universal element $\tilde u_{\mathrm{univ}}\in \tilde U(R(d))$ of Subsection 6.3 specializes to $\tilde u$, the truncated Barsotti–Tate group of level 1 of the pull back of $n^*_{R(d)/2R(d)}(\mathcal{A}_{\mathcal{N}_{k\mathrm{red}}})$ (i.e., of the abelian scheme $A'_{R(d)/2R(d)}$ of Subsection 6.4) via a dominant geometric point of $\mathrm{Spec}(R(d)/2R(d))$, is ordinary. Thus the ordinary locus of $\mathcal{N}_k$ specializes to the $k$-valued point of $\mathcal{N}_k$ defined by $y$. As $y$ is arbitrary, the Proposition follows. $\square$

**7.4. End of the proof of 1.4 (c).** From Propositions 7.1 and 7.3 we get that $\mathcal{N}_k^{\mathrm{s}}$ is a Zariski dense subscheme of $\mathcal{N}_k^{\mathrm{n}}$. From this and the open closed part of Proposition 7.2, we get that $\mathcal{N}_k^{\mathrm{s}}=\mathcal{N}_k^{\mathrm{n}}$. Therefore we have $\mathcal{N}^{\mathrm{s}}=\mathcal{N}^{\mathrm{n}}$, cf. Proposition 7.2. Thus Theorem 1.4 (c) holds. This ends the proof of the Main Theorem.

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Univ. of Arizona, Dept. of Mathematics, 617 N. Santa Rita,
P.O. Box 210089, Tucson, AZ-85721-0089, USA; e-mail: adrian@math.arizona.edu